On the axiomatizability of some first-order spatio-temporal theories

Sándor Vályi
College of Nyíregyháza

LR12

Abstract

- Spatio-temporal logic: variant of temporal logic
  time flow: one of the so-called causal relations on spacetime

- A first-order spatio-temporal theory over the rational order is recursively enumerable if and only if the dimension of spacetime does not exceed 2

- In the case of real co-ordinates: even dimension 2 does not permit recursive enumerability

Temporal logic

- propositional – modelling finite systems

- first-order – modelling arbitrary time-dynamical systems

- linear time vs. branching time

Axiomatizability

- means here just recursive enumerability
- first step in providing a complete reasoning system

Spatial dynamics of systems

- separate co-ordinates or separate modalities for space directions and time

- relativity: spacetime equipped by causality type relations

Axiomatizations concerning spacetime

- First-order: Robb – 1914 . . . Goldblatt – 1987
- Modal – propositional: Shehtman – 1983 and Goldblatt – 1980 (indep.)
- Second-order:
- FIrst-order, full equipped: Andréka, Németi et all. – about 1997

First-order temporal theories

- in most cases non-axiomatizable
- only for rational time flow (Reynolds – 1992)
- Situation for rational spacetime?

Temporal operator

A *temporal operator* is a triple $(\odot, k, \tau)$ where $\odot$ is a symbol, k is a positive integer and τ is a first-order formula in the signature $\mathcal{S}_n$ having a denumerably infinite set $\{t_0, t_1, \ldots\}$ of variables, a binary predicate symbol $\prec$, a finite set $\{P_1, P_2, \ldots P_n\}$ of unary predicate symbols and nothing else. Further requirement on τ is to contain exactly the only parameter t_0.

$\odot$ is the visual form of the operator, k is its arity while the role of τ is to describe the intended semantics of the operator. We will name the operators just by their first component, to avoid unneccessary complication of notations. Two examples of temporal operators are presented here:

$(\boxminus\!\!\!\rightarrow, 1, \forall t_1(t_0 \prec t_1 \rightarrow P_1(t_1))$ and

$(Until, 2, \exists t_1\{t_0 \prec t_1 \wedge P_2(t_1) \wedge \forall t_2[t_0 \prec t_2 \wedge t_2 \prec t_1 \rightarrow P_1(t_2)]\}\ )$.

First-order temporal syntax

TL_L^{Op}, based on a first-order signature L and a finite set Op of temporal operators, is the smallest set of formulae (on the appropriate alphabet) satisfying the following requirements:

- any atomic formula of L is an atomic temporal formula of TL_L^{Op},
- $(A \wedge B)$ and $\neg A$ are formulæ of TL_L^{Op} , if A and B belong to that set,
- $\forall x A$ is formula of TL_L^{Op} if A is a formula of TL_L^{Op} and x is a variable of L,
- $\odot(A_1, \ldots A_k)$ is a formula of TL_L^{Op} if $(\odot, k, \tau) \in Op$ for some table τ and $A_1, \ldots A_k$ are formulæ of TL_L^{Op}.

The set of terms of TL_L^{Op} coincides with the set of terms of pure L.

Syntax abbrev.

We assume the usual syntactic notions – as subformula, free and quantified variable, term substitution etc., modified in the adequate way – to be understood. We accept the usual abbreviations of first-order logic, as $(A \vee B)$, $(A \rightarrow B)$, $\exists x A$ etc., and use their well-known semantic properties without any extra remark. We provide here an example formula in TL_L^{Op}, where $Op = \{\boxminus\!\!\!\!\rightarrow, Until\}$ and signature L contains a unary predicate symbol p and a binary q: $\boxminus\!\!\!\!\rightarrow \forall x(p(x) \rightarrow \forall y\ Until(q(y,x), p(y)))$.

Time flow

A *time flow* is a non-empty partially ordered set $(T, \ll)$.

$(T, \ll)$ is the intended notion of time.

Temporal semantics

- interpretation of terms is time-independent
- valuation of variables time-independent
- only interpretation of predicate symbols is dynamical

Satisfaction

Just one item:

if $A = \odot(B_1, \ldots B_n)$ for a temporal operator $(\odot, n, \tau) \in Op$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff $\mathcal{B} \models \tau\binom{t_0}{t}$ where $\mathcal{B}$ is an interpretation for signature $\mathcal{S}_n$ (c.f. Def. 2.1) whose domain is T, further, $\prec^{\mathcal{B}} = \ll$ and the interpretation of P_i in $\mathcal{B}$ can be given as the subset of T consisting of time points where B_i holds, that is, $(P_i)^{\mathcal{B}} = \{s \in T : (T, \ll), \mathcal{I}, \Theta, s \Vdash B_i\}$ for any integer i between 1 and n.

Consequently, in the sense of the last definition, if $A = Until(B, C)$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff there exists an $s \in T$ such that $t \ll s$, $(T, \ll), \mathcal{I}, \Theta, s \Vdash B$ and for all $r \in T$ such that $t \ll r \ll s$, $(T, \ll), \mathcal{I}, \Theta, r \Vdash C$. Further, if $A = \boxminus\!\!\!\Rightarrow B$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff for all $s \in T$ such that $t \ll s$, $(T, \ll), \mathcal{I}, \Theta, s \Vdash B$.

Op-temporal theory

$\mathtt{Th}_L^{Op}(T, \ll)$ of time flow $(T, \ll)$ on signature L is the set of such closed TL_L^{Op}-formulæ A, that for any temporal interpretation $\mathcal{I}$, any $t \in T$ and any variable valuation Θ, $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ holds.

A set S of formulæ *axiomatizable* iff it is recursively enumerable.

Our actual operators

$$\mathsf{G} := (\boxminus\!\!\!\rightarrow, 1, \forall t_1(t_0 \prec t_1 \rightarrow P_1(t_1))$$

$$\mathsf{N} := (\boxed{\circ}\,, 1, \forall t_1(\neg\forall t_2(t_2 \ll t_0 \leftrightarrow t_2 \ll t_1) \rightarrow P_1(t_1)))$$

Let $n > 1$. We recall the definition for the function *Minkowskian distance* $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$. It is defined by $\mu((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = (x_1 - y_1)^2 - (x_2 - y_2)^2 - \ldots - (x_n - y_n)^2$. Further, for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $(x \blacktriangleleft y)$ for $\mu(x, y) > 0 \wedge x_1 < y_1$.

In special relativity theory, this relation is also known as *directed material or timelike causal accessibility* because it holds iff there is a possibility an event occuring in y to take a material (below-lightspeed) effect from an event in x. In this case we say also that y is inside of the upper lightcone of x.

Signature L_0 includes no equality symbol just one unary predicate symbol, namely, r.

Main result: Let $n > 2$. $\mathtt{Th}^{GN}_{L_0}(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable.

In contrast with: For any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}^{Op}_{L}(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable.

and with: Let $n \geq 2$. $\mathtt{Th}^{GN}_{L_0}(\mathbb{R}^n, \blacktriangleleft)$ is not axiomatizable.

Ideas of the proofs

Case $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 2)$
like non-axiomatizability proofs in first-order temporal logics
(e.g. Gabbay–Reynolds–Hodkinson's book)
except for the absence of binary or ternary relations
(signature is limited to only one unary predicate symbol without equality)
representation of predicates of more than one argument

Existing proofs of non-axiomatizability of *monadic* first-order temporal logic
Hughes–Creswell – it proves only non-decidability of some first-order modal logic –,
Hodkinson, Wolter, Zakaryashev, monOdic-first-order-temporal-logic-2000
Merz-1992)
cannot be followed directly more technical level, not a direct consequence of them

Ideas for rational spacetime

more difficult: prove the non-axiomatizability theorem for the rationals.
Extra difficulty:
In the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ one can express an equidistance formula (R. Goldblatt's book).
in $(\mathbb{Q}^n, \blacktriangleleft)$ it doesn't go through
new definition for spacelike betweenness which is also valid for $(\mathbb{Q}^n, \blacktriangleleft)$
describing a situation in $(\mathbb{Q}, \blacktriangleleft)$, by means of the just defined spacelike betweenness
makes possible the representation of the first-order theory of $(\mathbb{N}, +, *, =)$.

Definition of betweenness in the case of rational spacetime

- (i) $(x = y) \rightleftharpoons \forall z(z \blacktriangleleft x \leftrightarrow z \blacktriangleleft y)$,
- (ii) $(x \triangleleft y) \rightleftharpoons \forall z(y \blacktriangleleft z \rightarrow x \blacktriangleleft z) \wedge \neg x \blacktriangleleft y \wedge \neg x = y$,
- (iii) $\sigma(x, y) \rightleftharpoons x \not\blacktriangleleft y \ \wedge \ x \not\triangleleft y \ \wedge \ y \not\blacktriangleleft x \ \wedge \ y \not\triangleleft x \ \wedge \ y \neq x$,
- (iv) $\underline{\beta_\sigma}(x, z, y) \rightleftharpoons$
  $\sigma(x, y) \wedge \forall u(x \blacktriangleleft u \wedge y \blacktriangleleft u \rightarrow z \blacktriangleleft u) \wedge \forall u(u \blacktriangleleft x \wedge u \blacktriangleleft y \rightarrow u \blacktriangleleft z)$,
- (v) $\beta_\sigma(x, z, y) \rightleftharpoons \underline{\beta_\sigma}(x, z, y) \wedge x \neq z \wedge z \neq y$.

Thank you for your attention.
Happy birthday to István'