Lagrange geometry and a unified field theory

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Preliminaries

A parallelizable manifold is an n-dimensional C^∞ manifold M which admits n linearly independent global vector fields $\underset{i}{\lambda}$ $(i = 1, \ldots, n)$ on M. Such a space is also known in the literature as a teleparallel or Absolute Parallelism (AP-) space. Let $\underset{i}{\lambda}^\mu$ $(\mu = 1, \ldots, n)$ be the coordinate components of the i-th vector field $\underset{i}{\lambda}$. The Einstein summation convention is applied on both Latin (mesh) and Greek (world) indices, where all Latin indices are written in a lower position. The covariant components of $\underset{i}{\lambda}^\mu$ are given via the relations

$$\underset{i}{\lambda}^\mu \underset{i}{\lambda}_\nu = \delta^\mu_\nu, \qquad \underset{i}{\lambda}^\mu \underset{j}{\lambda}_\mu = \delta_{ij}. \tag{1}$$

The canonical connection is defined by

$$\Gamma^{\alpha}_{\mu\nu} := \underset{i}{\lambda}^{\alpha} \underset{i}{\lambda}_{\mu,\nu} \tag{2}$$

If "$|$" denotes covariant derivative with respect to the canonical connection, then

$$\lambda_{\mu|\nu} = 0, \qquad \lambda^{\mu}{}_{|\nu} = 0. \tag{3}$$

The above relation is known in the literature as the **AP-condition.**

Let

$$\Lambda^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} \tag{4}$$

denote the **torsion tensor** field of $\Gamma^{\alpha}_{\mu\nu}$. It is of particular importance to note that the AP-condition together with the commutation formula

$$\lambda^{\alpha}{}_{|\mu\nu} - \lambda^{\alpha}{}_{|\nu\mu} = \lambda^{\epsilon} R^{\alpha}_{\epsilon\nu\mu} + \lambda^{\alpha}{}_{|\epsilon} \Lambda^{\epsilon}_{\nu\mu}$$

forces the curvature tensor field $R^{\alpha}_{\mu\nu\sigma}$ of the canonical connection $\Gamma^{\alpha}_{\mu\nu}$ to vanish identically.

There are other three natural (built-in) connections which are non-flat. Namely, the **dual connection**

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\nu\mu}, \tag{5}$$

the **symmetric connection**

$$\widehat{\Gamma}^{\alpha}_{\mu\nu} := \frac{1}{2}(\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\mu}) = \Gamma^{\alpha}_{(\mu\nu)} \tag{6}$$

and the **Riemannian connection** (Christoffel symbols)

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} := \frac{1}{2}g^{\alpha\epsilon}(g_{\epsilon\nu,\mu} + g_{\epsilon\mu,\nu} - g_{\mu\nu,\epsilon}) \tag{7}$$

associated to the **metric structure** defined by

$$g_{\mu\nu} := \underset{i}{\lambda}_{\mu}\underset{i}{\lambda}_{\nu} \tag{8}$$

The **contortion tensor** field is defined by

$$\gamma^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} = \underset{i}{\lambda}{}^{\alpha} \underset{i}{\lambda}{}_{\mu \overset{o}{|} \nu}. \tag{9}$$

The **basic vector** field C_μ is defined by

$$C_\mu := \Lambda^{\alpha}_{\mu\alpha} = \gamma^{\alpha}_{\mu\alpha} \tag{10}$$

$$\Lambda^{\alpha}_{\mu\nu} = \gamma^{\alpha}_{\mu\nu} - \gamma^{\alpha}_{\nu\mu}, \tag{11}$$

$$\gamma_{\mu\nu\sigma} = \frac{1}{2}(\Lambda_{\mu\nu\sigma} + \Lambda_{\sigma\nu\mu} + \Lambda_{\nu\sigma\mu}). \tag{12}$$

The Generalized Field Theory (GFT) Starting with the Lagrangian

$$L := g^{\mu\nu} L_{\mu\nu} := g^{\mu\nu}(\Lambda^{\alpha}_{\epsilon\mu}\Lambda^{\epsilon}_{\alpha\nu} - C_\mu C_\nu), \tag{13}$$

the authors of the theory, using a certain variational technique, obtained the differential identity

$$E^{\mu}{}_{\widetilde{\nu|}\mu} = 0. \tag{14}$$

Regarding the above identity as representing a certain **conservation law**, the field equations of the GFT are taken to be

$$E_{\mu\nu} = 0; \tag{15}$$

where

$$\begin{aligned} E_{\mu\nu} := {} & g_{\mu\nu} L - 2L_{\mu\nu} - 2(C_\mu C_\nu - C_{\nu|\mu}) + 2g_{\mu\nu}(C^\epsilon C_\epsilon - C^{\epsilon}{}_{|\epsilon}) \\ & - 2(C^\epsilon \Lambda_{\mu\epsilon\nu} + g^{\epsilon\alpha}\Lambda_{\mu\nu\alpha|\epsilon}). \end{aligned} \tag{16}$$

Considering the **symmetric part** of (16) and after some intricate calculations, the Einstein field equations are found to be

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \tag{17}$$

in which the **energy-momentum tensor** $T_{\mu\nu}$ is expressed in terms of the fundamental second order symmetric tensor fields of Table 1.2 in the form

$$T_{\mu\nu} := \frac{1}{2} g_{\mu\nu}(\sigma - \varpi) - (\sigma_{\mu\nu} - \varpi_{\mu\nu}), \tag{18}$$

where σ and ϖ are the trace of $\sigma_{\mu\nu}$ and $\varpi_{\mu\nu}$ respectively. Moreover, according to (17), $T_{\mu\nu}$ satisfies the **conservation law**

$$T^{\mu\nu}{}_{\overset{o}{|}\mu} = 0. \tag{19}$$

On the other hand, considering the **skew-symmetric part** of (16), it is deduced that the **electromagnetic field** is expressed as the **curl** of the basic vector field C_μ, namely,

$$F_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu} \tag{20}$$

Moreover, it is given in terms of the fundamental second order skew-symmetric tensor fields of Table 1.2 in the form

$$F_{\mu\nu} := \gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu}, \tag{21}$$

In view of (20), $F_{\mu\nu}$ satisfies the (generalized) second **Maxwell's equation**

$$F_{\mu\nu\overset{o}{|}\sigma} + F_{\nu\sigma\overset{o}{|}\mu} + F_{\sigma\mu\overset{o}{|}\nu} = F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0. \tag{22}$$

Finally, the **tensor density** $\mathcal{F}^{\mu\nu} = |\lambda| F^{\mu\nu}$, $|\lambda| := det|\lambda_\beta|$, is introduced, from which the vector density

$$\mathcal{J}^{\mu} := \mathcal{F}^{\mu\nu}{}_{,\nu} \tag{23}$$

is defined. It satisfies the **conservation law**

$$\mathcal{J}^{\mu}{}_{,\mu} = \mathcal{J}^{\mu}{}_{\overset{o}{|\mu}} = 0. \tag{24}$$

$\mathcal{J}_\mu$ is interpreted as the **current density**.

The geometry of the tangent bundle:

Let M be a paracompact manifold of dimension n of class C^∞. Let $\pi : TM \to M$ be its tangent bundle. If $(U,\ x^\mu)$ is a local chart on M, then $(\pi^{-1}(U), (x^\mu,\ y^a))$ is the corresponding local chart on TM. The coordinate transformation on TM is given by:

$$x^{\mu'} = x^{\mu'}(x^\nu), \ \ y^{a'} = p_a^{a'} y^a,$$

$\mu = 1, \ldots, n;\ a = 1, \ldots, n;\ p_a^{a'} = \frac{\partial y^{a'}}{\partial y^a} = \frac{\partial x^{a'}}{\partial x^a}$ and $\det(p_a^{a'}) \neq 0$.

The paracompactness of M ensures the existence of a nonlinear connection N on TM with coefficients $N^a_\alpha(x,y)$. The transformation formula for the coefficients N^a_α is given by

$$N^{a'}_{\alpha'} = p^{a'}_a p^\alpha_{\alpha'} N^a_\alpha + p^{a'}_a p^a_{c'\alpha'} y^{c'}, \tag{25}$$

The nonlinear connection leads to the direct sum decomposition

$$T_u(TM) = H_u(TM) \oplus V_u(TM), \;\; \forall u \in TM \setminus \{0\}, \tag{26}$$

where $V_u(TM)$ is the vertical space at u with local basis $\dot{\partial}_a := \frac{\partial}{\partial y^a}$ and $H_u(TM)$ is the horizontal space at u, associated with N, supplementary to $V_u(TM)$, with local bases $\delta_\mu := \partial_\mu - N^a_\mu \, \dot{\partial}_a$.

Definition . *The curvature of a nonlinear connection is given by*

$$R^a_{\mu\nu} := \delta_\nu N^a_\mu - \delta_\mu N^a_\nu \tag{27}$$

Definition . *A nonlinear connection* N^a_μ *is said to be integrable if* $R^a_{\mu\nu} = 0$.

Definition . *A d-connection D on TM is a linear connection on TM which preserves by parallelism the horizontal and vertical distribution: if Y is a horizontal (vertical) vector field, then $D_X Y$ is a horizontal (vertical) vector field, for all $X \in \mathfrak{X}(TM)$.*

The coefficients of a d-connection $D = (\Gamma^{\alpha}_{\mu\nu},\ \Gamma^{a}_{b\nu},\ C^{\alpha}_{\mu c},\ C^{a}_{bc})$ are defined by

$$
\begin{aligned}
D_{\delta_\nu}\delta_\mu &=: \Gamma^{\alpha}_{\mu\nu}\delta_\alpha, \qquad & D_{\delta_\nu}\dot{\partial}_b &=: \Gamma^{a}_{b\nu}\dot{\partial}_a; \\
D_{\dot{\partial}_c}\delta_\mu &=: C^{\alpha}_{\mu c}\delta_\alpha, \qquad & D_{\dot{\partial}_c}\dot{\partial}_b &=: C^{a}_{bc}\dot{\partial}_a.
\end{aligned}
\tag{28}
$$

Definition . *An* hv*-metric on* TM *is a covariant* d*-tensor field* $\mathcal{G} := h\mathcal{G} + v\mathcal{G}$ *on* TM*, where* $h\mathcal{G} := g_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta},\ v\mathcal{G} := g_{ab}\, \delta y^{a} \otimes \delta y^{b}$ *such that:*

$$g_{\alpha\beta} = g_{\beta\alpha}, \ \ det(g_{\alpha\beta}) \neq 0; \quad g_{ab} = g_{ba}, \ \ det(g_{ab}) \neq 0.$$

Definition . *A* d*-connection* D *on* TM *is said to be metric or compatible with the metric* $\mathcal{G}$ *if*

$$g_{\alpha\beta|\mu} = g_{\alpha\beta||c} = g_{ab|\mu} = g_{ab||c} = 0. \tag{29}$$

Theorem . *For a given* hv*-metric on* TM*, there exists a unique metric* d*-connection* $\overset{\circ}{D} = (\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}, \overset{\circ}{\Gamma}{}^{a}_{b\nu}, \overset{\circ}{C}{}^{\alpha}_{\mu c}, \overset{\circ}{C}{}^{a}_{bc})$ *on* TM *with the properties that*

(a) $\overset{\circ}{\Gamma}{}^{a}_{b\nu} := \dot{\partial}_b N^a_\nu + \frac{1}{2} g^{ac}(\delta_\nu g_{bc} - g_{dc}\,\dot{\partial}_b N^d_\nu - g_{bd}\,\dot{\partial}_c N^d_\nu), \quad \overset{\circ}{C}{}^{\alpha}_{\mu c} := \frac{1}{2} g^{\alpha\epsilon}\dot{\partial}_c g_{\mu\epsilon}.$

(b) $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} := \frac{1}{2} g^{\alpha\epsilon}(\delta_\mu g_{\epsilon\nu} + \delta_\nu g_{\epsilon\mu} - \delta_\epsilon g_{\mu\nu}), \quad \overset{\circ}{C}{}^{a}_{bc} := \frac{1}{2} g^{ad}(\dot{\partial}_b g_{dc} + \dot{\partial}_c g_{db} - \dot{\partial}_d g_{bc})$

We call the connection $\overset{\circ}{D}$ the **natural metric** d-connection.

EAP-space

We assume that $\underset{i}{\lambda}$, $i = 1, \ldots, n$, are n vector fields **globally** defined on TM. In the adapted basis $(\delta_\alpha, \dot{\partial}_a)$, we have $\underset{i}{\lambda} = h\underset{i}{\lambda} + v\underset{i}{\lambda} = \underset{i}{\lambda}^\alpha \delta_\alpha + \underset{i}{\lambda}^a \dot{\partial}_a$. *We further assume that the n horizontal vector fields $h\underset{i}{\lambda}$ and the n vertical vector fields $v\underset{i}{\lambda}$ are separately linearly independent* so that

$$\underset{i}{\lambda}^\alpha \underset{i}{\lambda}_\beta = \delta^\alpha_\beta, \quad \underset{i}{\lambda}^\alpha \underset{j}{\lambda}_\alpha = \delta_{ij}; \quad \underset{i}{\lambda}^a \underset{i}{\lambda}_b = \delta^a_b, \quad \underset{i}{\lambda}^a \underset{j}{\lambda}_a = \delta_{ij}, \qquad (30)$$

where $(\underset{i}{\lambda}_\alpha)$ and $(\underset{i}{\lambda}_a)$ denote the inverse matrices of $(\underset{i}{\lambda}^\alpha)$ and $(\underset{i}{\lambda}^a)$ respectively. We refer to the vector fields $\underset{i}{\lambda}$ as the **fundamental vector fields**.

In this case, the metric is given by:

$$g_{\alpha\beta} := \underset{i}{\lambda}_{\alpha} \underset{i}{\lambda}_{\beta}, \quad g_{ab} := \underset{i}{\lambda}_{a} \underset{i}{\lambda}_{b}. \tag{31}$$

The inverse of the matrices $(g_{\alpha\beta})$ and (g_{ab}) are given by $(g^{\alpha\beta})$ and (g^{ab}) respectively, where

$$g^{\alpha\beta} = \underset{i}{\lambda}^{\alpha} \underset{i}{\lambda}^{\beta}, \quad g^{ab} = \underset{i}{\lambda}^{a} \underset{i}{\lambda}^{b}. \tag{32}$$

Theorem . *The* d*-connection* $D = (\Gamma^{\alpha}_{\mu\nu},\ \Gamma^{a}_{b\nu},\ C^{\alpha}_{\mu c},\ C^{a}_{bc})$ *defined by*

$$\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}(\delta_{\nu}\underset{i}{\lambda}_{\mu}), \Gamma^{a}_{b\nu} = \underset{i}{\lambda}^{a}(\delta_{\nu}\underset{i}{\lambda}_{b}); C^{\alpha}_{\mu c} = \underset{i}{\lambda}^{\alpha}(\dot{\partial}_{c}\underset{i}{\lambda}_{\mu}), C^{a}_{bc} = \underset{i}{\lambda}^{a}(\dot{\partial}_{c}\underset{i}{\lambda}_{b}) \tag{33}$$

satisfies the AP-condition

$$\lambda^{\alpha}{}_{|\mu} = \lambda^{\alpha}{}_{||c} = \lambda^{a}{}_{|\mu} = \lambda^{a}{}_{||c} = 0. \tag{34}$$

Consequently, D *is a metric* d*-connection.*

This d-connection is referred to as the **canonical** d-connection.

Definition . *The torsion tensor field* $\mathbf{T} = (\Lambda^{\alpha}_{\mu\,\nu}, R^{a}_{\mu\nu}, C^{\alpha}_{\mu c}, P^{a}_{\mu c}, T^{a}_{bc})$ *of the canonical* d*-connection is referred to as the torsion of the EAP-space.*

Definition . *In the adapted basis* $(\delta_\mu,\ \dot{\partial}_a)$*, the contortion tensor* $\mathbf{C}$ *is characterized by the* d*-tensor fields with local coefficients* $(\gamma^{\alpha}_{\mu\nu}, \gamma^{a}_{b\nu}, \gamma^{\alpha}_{\mu c}, \gamma^{a}_{bc})$ *defined by:*

$$\begin{aligned} \gamma^{\alpha}_{\mu\nu} &:= \Gamma^{\alpha}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}, \quad \gamma^{a}_{b\mu} := \Gamma^{a}_{b\mu} - \overset{\circ}{\Gamma}{}^{a}_{b\mu}; \\ \gamma^{\alpha}_{\mu c} &:= C^{\alpha}_{\mu c} - \overset{\circ}{C}{}^{\alpha}_{\mu c}, \quad \gamma^{a}_{bc} := C^{a}_{bc} - \overset{\circ}{C}{}^{a}_{bc}. \end{aligned} \tag{35}$$

Definition . *The basic vector* $\mathbf{B} = (C_\mu, C_a)$ *is given by*

$$C_\mu := \Lambda^{\alpha}_{\mu\alpha} = \gamma^{\alpha}_{\mu\alpha}, \qquad C_b := T^{a}_{ba} = \gamma^{a}_{ba} \tag{36}$$

Definition . *Let* $D = (\Gamma^{\alpha}_{\mu\nu}, \Gamma^{a}_{b\mu}, C^{\alpha}_{\mu c}, C^{a}_{bc})$ *be the canonical* d-*connection.*

(a) *The* **dual** d-*connection* $\widetilde{D} = (\tilde{\Gamma}^{\alpha}_{\mu\nu}, \tilde{\Gamma}^{a}_{b\mu}, \tilde{C}^{\alpha}_{\mu c}, \tilde{C}^{a}_{bc})$ *is defined by*

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\nu\mu}, \tilde{\Gamma}^{a}_{b\mu} := \Gamma^{a}_{b\mu}; \tilde{C}^{\alpha}_{\mu c} := C^{\alpha}_{\mu c}, \tilde{C}^{a}_{bc} := C^{a}_{cb}. \tag{37}$$

(b) *The* **symmetric** d-*connection* $\widehat{D} = (\hat{\Gamma}^{\alpha}_{\mu\nu}, \hat{\Gamma}^{a}_{b\mu}, \hat{C}^{\alpha}_{\mu c}, \hat{C}^{a}_{bc})$ *is defined by*

$$\begin{aligned} \hat{\Gamma}^{\alpha}_{\mu\nu} &:= \frac{1}{2}(\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\mu}), \quad \hat{\Gamma}^{a}_{b\mu} := \Gamma^{a}_{b\mu}; \\ \hat{C}^{\alpha}_{\mu c} &:= C^{\alpha}_{\mu c}, \qquad \hat{C}^{a}_{bc} := \frac{1}{2}(C^{a}_{bc} + C^{a}_{cb}). \end{aligned} \tag{38}$$

The Cartan and Berwald-type cases

(1) The Cartan-type case

Assume that the canonical d-connection D is of Cartan-type: ${y^a}_{|\mu} = 0, \quad {y^a}_{||c} = \delta^a_c$. Consequently,

(a) The nonlinear connection N^a_μ is expressed in the form $N^a_\mu = y^b(\underset{i}{\lambda}{}^a \partial_\mu \underset{i}{\lambda}{}_b)$.

(b) The nonlinear connection N^a_μ is integrable: $R^a_{\mu\nu} = 0$. Moreover, $T^a_{bc} = \gamma^a_{bc} = 0$.

(2) The Berwald-type case Assume that D is of Berwald-type:

$\dot{\partial}_b N^a_\mu = \Gamma^a_{b\mu}$; $C^\alpha_{\mu c} = 0$. Consequently,

(a) λ_μ are functions of the positional argument x only. Consequently, so are $g_{\mu\nu}$.

(b) $\Lambda^\alpha_{\mu\nu}$, $\gamma^\alpha_{\mu\nu}$ and C_μ are functions of the positional argument x only.

Theorem . *Assume that D is both of Cartan- and Berwald-type. Then*

(a) *The hh-coefficients of the four defined d-connections are functions of positional argument only and are identical to the coefficients of the corresponding connections in the conventional AP-space.*

(b) *The torsion and the contortion tensor fields of the EAP-space are functions of positional argument only and are given by*

$$\mathbf{T} = (\Lambda^{\alpha}_{\mu\nu}, 0, 0, 0, 0); \quad \mathbf{C} = (\gamma^{\alpha}_{\mu\nu}, 0, 0, 0)$$

Unified field equations

Unified horizontal field equations

Let

$$\mathcal{H} = |\lambda| g^{\mu\nu} H_{\mu\nu},$$

where

$$H_{\mu\nu} := \Lambda^{\alpha}_{\epsilon\mu} \Lambda^{\epsilon}_{\alpha\nu} - C_{\mu} C_{\nu}. \tag{39}$$

The Euler-Lagrange equations for this Lagrangian are given by

$$\frac{\delta \mathcal{H}}{\delta \lambda_{\beta}} := \frac{\partial \mathcal{H}}{\partial \lambda_{\beta}} - \frac{\partial}{\partial x^{\gamma}} \left(\frac{\partial \mathcal{H}}{\partial \lambda_{\beta,\gamma}} \right) - \frac{\partial}{\partial y^{a}} \left(\frac{\partial \mathcal{H}}{\partial \lambda_{\beta;a}} \right) = 0. \tag{40}$$

The **unified horizontal field** equations in the context of the EAP-geometry have the form

$$
\begin{aligned}
0 = {} & g_{\mu\nu}H - 2H_{\mu\nu} - 2C_\mu C_\nu - 2g_{\mu\nu}(C^{\epsilon}{}_{|\epsilon} - C^{\epsilon}C_{\epsilon}) - 2C^{\epsilon}\Lambda_{\mu\epsilon\nu} + 2C_{\nu|\mu} \\
& - \; 2g^{\epsilon\alpha}\Lambda_{\mu\nu\alpha|\epsilon} - 2N^{a}_{\epsilon;a}(\Lambda^{\epsilon}{}_{\nu\mu} - \Lambda_{\mu\nu}{}^{\epsilon}) + 2g_{\mu\nu}C^{\epsilon}N^{a}_{\epsilon;a} - 2C_\mu N^{a}_{\nu;a} \\
& + 2\,\mathfrak{S}_{\mu,\nu,\epsilon}C^{\epsilon}_{\mu a}R^{a}_{\nu\epsilon}.
\end{aligned}
\tag{41}
$$

Unified vertical field equations

Let

$$\mathcal{V} := ||\lambda|| g^{ab} V_{ab},$$

where

$$V_{ab} := T^{d}_{ea} T^{e}_{db} - C_a C_b. \tag{42}$$

The Euler-Lagrange equations in this case reduce to

$$\frac{\partial \mathcal{V}}{\partial \lambda_b} - \frac{\partial}{\partial y^e}\left(\frac{\partial \mathcal{V}}{\partial \lambda_{b;\,e}}\right) = 0. \tag{43}$$

The **unified vertical field** equations in the context of the EAP-geometry have the form

$$0 = g_{ab}V - 2V_{ab} - 2g_{ab}({C^{e}}_{||e} - C^{e}C_{e}) - 2C_{a}C_{b} - 2C^{e}T_{aeb} + 2C_{b||a} - 2g^{de}T_{abe||d}. \tag{44}$$

Physical consequences

Splitting of the horizontal field equations

Symmetric part: Considering the symmetric part of (41), we get

$$0 = (g_{\mu\nu}\overset{\circ}{\mathcal{R}} - 2\overset{\circ}{R}_{(\mu\nu)}) + g_{\mu\nu}(\sigma - h - Q) - 2(\sigma_{\mu\nu} - h_{\mu\nu} - Q_{(\mu\nu)}) \\ + N^{\beta}(\Lambda_{\mu\nu\beta} + \Lambda_{\nu\mu\beta}) + 2g_{\mu\nu}C^{\beta}N_{\beta} - (C_{\mu}N_{\nu} + C_{\nu}N_{\mu}), \tag{45}$$

which represents the **symmetric part of the horizontal unified field equations**.

Consequently,

$$\overset{\circ}{R}_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{\mathcal{R}} = T_{(\mu\nu)}, \tag{46}$$

$$T_{(\mu\nu)} = \frac{1}{2} g_{\mu\nu}(\sigma - h - Q + 2Z) - (\sigma_{\mu\nu} - h_{\mu\nu} - Q_{(\mu\nu)}) + (\frac{1}{2} N_\beta \Omega^\beta_{\mu\nu} - Z_{(\mu\nu)}). \tag{47}$$

According to (46), $T_{(\mu\nu)}$ may be interpreted as the **geometric energy momentum tensor**.

Skew-symmetric part: Considering the skew-symmetric part of (41), we get

$$0 = 2\{(\gamma_{\mu\nu} - \epsilon_{\mu\nu} - \xi_{\mu\nu} + N_\beta \Lambda^\beta_{\mu\nu}) + (M_{[\mu\nu]} - Z_{[\mu\nu]})\} + 3\mathfrak{S}_{\mu,\nu,\epsilon} C^\epsilon_{\nu a} R^a_{\epsilon\mu}. \tag{48}$$

Consequently, if

$$\begin{aligned} F_{\mu\nu} := (\gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu}) + N^\beta(\gamma_{\mu\nu\beta} + \Lambda_{\beta\mu\nu}) + \\ (\frac{1}{2} N^\beta \Lambda_{\beta\mu\nu} - Z_{[\mu\nu]}) + \frac{3}{2} \mathfrak{S}_{\mu,\nu,\epsilon} C^\epsilon_{\nu a} R^a_{\epsilon\mu}, \end{aligned} \tag{49}$$

then

$$F_{\mu\nu} = \delta_\nu C_\mu - \delta_\mu C_\nu, \tag{50}$$

$$\mathfrak{S}_{\mu,\nu,\sigma} \, F_{\mu\nu \overset{o}{|} \sigma} = - \, \mathfrak{S}_{\mu,\nu,\sigma} \, R^{a}_{\mu\nu} \dot{\partial}_a C_\sigma. \tag{51}$$

Accordingly, if $F_{\mu\nu}$ is interpreted as the **horizontal geometric electromagnetic field**, then (51) represents the **horizontal generalized Maxwell's equations** and, in view of (50), C_μ is the **horizontal geometric electromagnetic potential**.

Now, let

$$J^{\mu} := F^{\mu\nu}{}_{\overset{o}{|}\nu}. \tag{52}$$

Then

$$J^{\mu}{}_{\overset{o}{|}\mu} = \frac{1}{2}\{F^{\epsilon\mu}(\overset{\circ}{R}_{\mu\epsilon} - \overset{\circ}{R}_{\epsilon\mu}) + R^{a}_{\mu\nu}F^{\mu\nu}{}_{\overset{o}{||}a}\}. \tag{53}$$

In the case where the nonlinear connection N^{α}_{μ} is integrable, (53) can be viewed as a generalized conservation law and J^{μ} as the **geometric horizontal current density**.

Splitting of the vertical field equations

Symmetric part: Considering the skew-symmetric part of (44), we get

$$0 = E_{(ab)} := (g_{ab}\mathring{\mathcal{S}} - 2\mathring{S}_{ab}) + g_{ab}(\bar{\sigma} - \bar{h}) - 2(\sigma_{ab} - h_{ab}),$$

so that

$$\mathring{S}_{ab} - \frac{1}{2} h_{ab}\mathring{\mathcal{S}} = T_{ab}, \tag{54}$$

$$T_{ab} := \frac{1}{2} g_{ab}(\bar{\sigma} - \bar{h}) - (\sigma_{ab} - h_{ab}). \tag{55}$$

$$T^{a}{}_{b\overset{o}{||}a} = 0. \tag{56}$$

Consequently, in view of (54) and (56), T_{ab} could be interpreted as the **vertical geometric energy-momentum tensor** for both matter and electromagnetism.

Skew-symmetric part: Considering the skew-symmetric part of (44), we conclude that if

$$F_{ab} := \gamma_{ab} - \xi_{ab} + \eta_{ab}, \tag{57}$$

then F_{ab} is the **vertical geometric electromagnetic field** and, in view of the relation

$$F_{ab} = \dot{\partial}_b C_a - \dot{\partial}_a C_b, \tag{58}$$

C_a may be interpreted as the **vertical geometric electromagnetic potential**. Moreover, we obtain the **vertical** generalized **Maxwell's equations**

$$\mathfrak{S}_{a,b,c} F_{ab\overset{o}{||}c} = 0. \tag{59}$$

Finally, if we set

$$J^a := F^{ab}{}_{\overset{o}{|}b} \tag{60}$$

then, similar to (53), J^a satisfies the **conservation law**

$$J^a{}_{\overset{o}{|}a} = 0. \tag{61}$$

Hence, J^a represents the **vertical geometric current density**.

Important special cases

Integrability condition

The nonlinear connection is integrable: $R^{a}_{\mu\nu} = 0$. Then, we have **Energy momentum tensor**:

$$T_{\mu\nu} := \{\frac{1}{2} g_{\mu\nu}(\sigma - h) + (h_{\mu\nu} - \sigma_{\mu\nu})\} + g_{\mu\nu} Z + A_{(\mu\nu)} \quad (62)$$

Electromagnetic field:

$$F_{\mu\nu} := (\gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu}) + N^{\beta}(\gamma_{\mu\nu\beta} + \Lambda_{\beta\mu\nu}) + A_{[\mu\nu]}. \quad (63)$$

Conservation law: $T^{\mu}{}_{\nu \overset{o}{|} \mu} = 0.$ (64)

Maxwell's equations: $\mathfrak{S}_{\mu,\nu,\sigma} F_{\mu\nu \overset{o}{|} \sigma} = 0$ (65)

Conservation law: $J^{\mu}{}_{\overset{o}{|} \mu} = 0.$ (66)

The Cartan-type case

The Cartan-type case can be regarded as a special case of the integrability case, obtained by setting $N^a_\mu = y^b(\underset{i}{\lambda}{}^a \partial_\mu \underset{i}{\lambda}{}_b)$ and $R^a_{\mu\nu} = 0$ (among other things). Accordingly, relations (62) to (66) remain valid under the Cartan type condition. On the other hand, there are no vertical field equations (all vertical objects of Table 1 vanish). The advantage in this case, is that the nonlinear connection, consequently, all geometric objects considered, are expressed explicitily in terms of the fundamental vector fields λ's.

The Berwald type case

The **horizontal** field equations in this case are given by

$$0 = E_{\mu\nu} := g_{\mu\nu}H - 2H_{\mu\nu} - 2(C_\mu C_\nu - C_{\nu|\mu}) + 2g_{\mu\nu}(C^\epsilon C_\epsilon - C^\epsilon{}_{|\epsilon}) \\ - 2(C^\epsilon \Lambda_{\mu\epsilon\nu} + g^{\epsilon\alpha}\Lambda_{\mu\nu\alpha|\epsilon}), \tag{67}$$

which are **identical** in form to the field equations of the GFT. Moreover, all geometrical objects involved in (67) are functions of the positional argument x only.

In this case, we have

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{\mathcal{R}} = T_{\mu\nu};$$

Energy momentum tensor: $T_{\mu\nu} := \frac{1}{2} g_{\mu\nu}(\sigma - h) - (\sigma_{\mu\nu} - h_{\mu\nu}),$ (68)

Conservation Law: $T^{\mu}{}_{\nu \overset{o}{|} \mu} = 0.$ (69)

Electomagnetic field: $F_{\mu\nu} := \gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu} = \partial_\nu C_\mu - \partial_\mu C_\nu.$

Maxwell's equations: $\mathfrak{S}_{\mu,\nu,\sigma} F_{\mu\nu \overset{o}{|} \sigma} = 0,$ (70)

Conservation Law: $J^{\mu}{}_{\overset{o}{|} \mu} = 0.$ (71)

The Cartan-Berwald case (Recovering the GFT)

We finally assume that the canonical d-connection is both of Berwald- and Cartan-type. In this case, the horizontal field equations are given by (67), whereas the vertical field equations clearly disappear. Moreover, relations (68) to (71) hold. Consequently, the field equations obtained coincide in form and content with those of the GFT. This is the typical case in which the GFT is naturally retrieved.

Concluding remarks

- We have constructed a unified field theory in the framework of EAP-geometry. The formulated theory is a generalization of the GFT, in which the chosen Lagrangians are the horizontal and vertical analogues of the Lagrangian used in the construction of the GFT. Five different interesting cases for the horizontal field equations have been singled out. The most general is derived under the mere assumption that the nonlinear connection is independent of the horizontal fundamental vector fields. From this, follows both the Integrability case and the Cartan case. The Berwald case is also deduced independently. Finally, under the Cartan-Berwald condition, the constructed field equations are shown to coincide with the GFT.

- Our constructed field theory is a pure geometrical attempt to unify gravity and electromagnetism. The theory is manifestly covariant. The underlying geometry of the theory is the EAP-geometry. The symmetric part represents gravitation, while the anti-symmetric part represents electromagnetism. Finally, all **physical** objects involved are expressed in terms of the **fundamental** tensors of the EAP-space together with the **nonlinear connection** N (and its curvature).

Q/A

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