Neat embeddings as adjoint situations

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Abstract

We show that certain properties of dimension complemented cylindric algebras, concerning neat embeddings, do not generalize much further. We show that certain reducts of polyadic algebras have the superamalgamation property. We consider those reducts of polyadic algebras where cylindrifiers are finite and all substitutions are available. This is an interesting stage between cylindric and polyadic algebras, which we show, share some positive properties of polyadic algebras. We also give a nice categorical formulation of an equivalence between such classes (which are varieties) of different infinite dimensions.

We recall from [1] Theorems 2.6.67, 2.6.71-72.

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Theorem 2.6.67

Assume that $\mathfrak{A} \in \mathbf{CA}_\alpha$, $\mathfrak{B} \in \mathbf{CA}_\beta$ and $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$. Then we have:

1. $\mathfrak{Nr}_\alpha\mathfrak{B}$ is a also a generating subreduct of $\mathfrak{B}$, and the same applies to every $\mathbf{CA}_\alpha$ $\mathfrak{C}$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$.
2. If, in addition, $\alpha \geq \omega$ and $\mathfrak{A} \in \mathbf{Dc}_\alpha$, then $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$, and hence $\mathfrak{Nr}_\alpha\mathfrak{B}$ is then the unique α dimensional generating subreduct of $\mathfrak{B}$.

We recall from [1] Theorems 2.6.67, 2.6.71-72.

Theorem 2.6.71

Assume $\alpha \geq \omega$, $\mathfrak{A} \in \mathbf{Dc}_\alpha$, $\mathfrak{A}$ is a generating subreduct of a **CA** $\mathfrak{B}$, and $I \in Il\mathfrak{B}$. Then $I = \mathfrak{Ig}^{\mathfrak{B}}(I \cap A)$.

We recall from [1] Theorems 2.6.67, 2.6.71-72.

Theorem 2.6.72

Assume $\beta \geq \alpha \geq \omega$, $\mathfrak{A} \in \mathbf{Dc}_\alpha$, $\mathfrak{B}, \mathfrak{B}' \in \mathbf{CA}_\beta$ and $\mathfrak{A}$ is a generating subreduct of both $\mathfrak{B}$ and $\mathfrak{B}'$. Then there is an $h \in Is(\mathfrak{B}, \mathfrak{B}')$ such that $A \upharpoonright h = A \upharpoonright Id$.

We quote Henkin, Monk and Tarski in [1]:“It will be shown in Part II that for each α, β such that $\beta \geq \alpha \geq \omega$ there is a $\mathbf{CA}_\alpha$ $\mathfrak{A}$ and a $\mathbf{CA}_\beta$ $\mathfrak{B}$ such that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$ different from $\mathfrak{Nr}_\alpha\mathfrak{B}$;

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And we quote Henkin and Monk in the introduction of [2]: “Throughout Part I various “promises" were made about material which would be found in Part II. These are located in this volume at the appropriate places, with the following exceptions, which mainly concern results whose proofs could not be reconstructed."

And we quote Henkin and Monk in the introduction of [2]: “Throughout Part I various “promises" were made about material which would be found in Part II. These are located in this volume at the appropriate places, with the following exceptions, which mainly concern results whose proofs could not be reconstructed."It turns out that these are 5 (unfulfilled) items, cf. [2]. Item (5) in op.cit. reads:

And we quote Henkin and Monk in the introduction of [2]: “Throughout Part I various “promises" were made about material which would be found in Part II. These are located in this volume at the appropriate places, with the following exceptions, which mainly concern results whose proofs could not be reconstructed."It turns out that these are 5 (unfulfilled) items, cf. [2]. Item (5) in op.cit. reads:
“Cf. Part 1 page 426. We do not know whether, if $\omega \leq \alpha < \beta$, there is a $\mathbf{CA}_\alpha$ $\mathfrak{A}$ and a $\mathbf{CA}_\beta$ $\mathfrak{B}$ such that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$ different from $\mathfrak{Nr}_\alpha\mathfrak{B}$."

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To the best of our knowledge counterexamples to generalizations of 2.6.71-72 in [1] are also unknown. We now show that in the above quoted theorems, $\mathbf{Dc}_\alpha$ cannot be replaced by $\mathbf{RCA}_\alpha$ confirming what seems to have been a conjecture of Tarski's, the proof of which could not be reconstructed by his co-authors Henkin and Monk.

To the best of our knowledge counterexamples to generalizations of 2.6.71-72 in [1] are also unknown. We now show that in the above quoted theorems, $\mathbf{Dc}_\alpha$ cannot be replaced by $\mathbf{RCA}_\alpha$ confirming what seems to have been a conjecture of Tarski's, the proof of which could not be reconstructed by his co-authors Henkin and Monk. In what follows, we use the notation of the monograph [1], often without warning, with the following exception. We write $f \upharpoonright A$ instead of $A \upharpoonright f$ to denote the restriction of f to A.

Lemma

If $\alpha < \beta$ are any ordinals and $L \subseteq \mathbf{CA}_\beta$, then, in the sequence of conditions (1) - (5) below, (1) - (4) implies the immediately following one:

(1) *For any $\mathfrak{A} \in L$ and $\mathfrak{B} \in \mathbf{CA}_\beta$ with $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, for all $X \subseteq A$ we have $\mathfrak{S}g^{\mathfrak{A}}X = \mathfrak{Nr}_\alpha\mathfrak{S}g^{\mathfrak{B}}X$.*

(2) *For any $\mathfrak{A} \in L$ and $\mathfrak{B} \in \mathbf{CA}_\beta$ with $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, if $\mathfrak{S}g^{\mathfrak{B}}A = \mathfrak{B}$, then $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$.*

(3) *For any $\mathfrak{A} \in L$ and $\mathfrak{B} \in \mathbf{CA}_\beta$ with $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, if $\mathfrak{S}g^{\mathfrak{B}}A = \mathfrak{B}$, then for any ideal I of $\mathfrak{B}$, $\mathfrak{Ig}^{\mathfrak{B}}(A \cap I) = I$.*

Lemma

If $\alpha < \beta$ are any ordinals and $L \subseteq \mathbf{CA}_\beta$, then, in the sequence of conditions (1) - (5) below, (1) - (4) implies the immediately following one:

(4) *If whenever $\mathfrak{A} \in L$, there exists $x \in {}^{|A|}A$ such that if $\rho = \langle \Delta x_i : i < |A| \rangle$, $\mathfrak{D} = \mathfrak{Fr}^{\rho}_{|A|}\mathbf{CA}_\beta$ and $g_\xi = \xi / Cr^{\rho}_{|A|}\mathbf{CA}_\beta$, then $\mathfrak{Sg}^{\mathfrak{Rd}_\alpha \mathfrak{D}}\{g_\xi : \xi < |A|\} \in L$, then the following hold: For $\mathfrak{A}, \mathfrak{A}' \in L$, $\mathfrak{B}, \mathfrak{B}' \in \mathbf{CA}_\beta$ with embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{B}$ and $e_{A'} : \mathfrak{A}' \to \mathfrak{Nr}_\alpha \mathfrak{B}'$ such that $\mathfrak{Sg}^{\mathfrak{B}} e_A(A) = \mathfrak{B}$ and $\mathfrak{Sg}^{\mathfrak{B}'} e_{A'}(A) = \mathfrak{B}'$, whenever $i : \mathfrak{A} \longrightarrow \mathfrak{A}'$ is an isomorphism, then there exists an isomorphism $\bar{i} : \mathfrak{B} \longrightarrow \mathfrak{B}'$ such that $\bar{i} \circ e_A = e_{A'} \circ i$.*

Lemma

If $\alpha < \beta$ are any ordinals and $L \subseteq \mathbf{CA}_\beta$, then, in the sequence of conditions (1) - (5) below, (1) - (4) implies the immediately following one:

(5) *Assume that $\beta = \alpha + \omega$. Then L has the amalgamation property with respect to $\mathbf{RCA}_\alpha$. That is for all $\mathfrak{A}_0 \in L$, $\mathfrak{A}_1$ and $\mathfrak{A}_2 \in \mathbf{RCA}_\alpha$, and all monomorphisms i_1 and i_2 of $\mathfrak{A}_0$ into $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, there exists $\mathfrak{A} \in \mathbf{RCA}_\alpha$, a monomorphism m_1 from $\mathfrak{A}_1$ into $\mathfrak{A}$ and a monomorphism m_2 from $\mathfrak{A}_2$ into $\mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$.*

Proof. (1) implies (2) is trivial. ∎

Proof. Now we prove (2) implies (3). The proof is similar to [1] 2.6.71. From the premise that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$ we easily infer that $|\Delta x \sim \alpha| < \omega$ for all $x \in B$. We now have $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$. Now clearly $\mathfrak{Ig}^{\mathfrak{B}}(I \cap A) \subseteq I$. Conversely let $x \in I$. Then $\mathsf{c}_{(\Delta x \sim \alpha)}x$ is in $\mathfrak{Nr}_\alpha\mathfrak{B}$, hence in $\mathfrak{A}$. Therefore $\mathsf{c}_{(\Delta x \sim \alpha)}x \in A \cap I$. But $x \leq \mathsf{c}_{(\Delta x \sim \alpha)}x$, hence the required. ∎

Proof. We now prove (3) implies (4). The proof is a generalization of the proof of [1] 2.6.72. Let $\mathfrak{A}, \mathfrak{A}' \in L$, $\mathfrak{B}, \mathfrak{B}' \in \mathbf{CA}_\beta$ and assume that $e_A, e_{A'}$ are embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{Nr}_\alpha\mathfrak{B}, \mathfrak{Nr}_\alpha\mathfrak{B}'$, respectively, such that $\mathfrak{S}g^{\mathfrak{B}}(e_A(A)) = \mathfrak{B}$ and $\mathfrak{S}g^{\mathfrak{B}'}(e_{A'}(A')) = \mathfrak{B}'$, and let $i : \mathfrak{A} \longrightarrow \mathfrak{A}'$ be an isomorphism. We need to “lift" i to β dimensions. Let $\mu = |A|$. Let x be a bijection from μ onto A that satisfies the premise of (4). Let y be a bijection from μ onto A', such that $i(x_j) = y_j$ for all $j < \mu$. Let $\rho = \langle \Delta^{(\mathfrak{A})} x_j : j < \mu \rangle$, $\mathfrak{D} = \mathfrak{Fr}_\mu^{(\rho)}\mathbf{CA}_\beta$, $g_\xi = \xi / Cr_\mu^{(\rho)}\mathbf{CA}_\beta$ for all $\xi < \mu$ and $\mathfrak{C} = \mathfrak{S}g^{\mathfrak{Rd}_\alpha\mathfrak{D}}\{g_\xi : \xi < \mu\}$. Then
$\mathfrak{C} \subseteq \mathfrak{Nr}_\alpha\mathfrak{D}$, C generates $\mathfrak{D}$ and by hypothesis $\mathfrak{C} \in L$. ■

Proof. There exist $f \in Hom(\mathfrak{D}, \mathfrak{B})$ and $f' \in Hom(\mathfrak{D}, \mathfrak{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and $f'(g_\xi) = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that f and f' are both onto. We now have $e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \upharpoonright \mathfrak{C}) = f \upharpoonright \mathfrak{C}$. Therefore $Kerf' \cap \mathfrak{C} = Kerf \cap \mathfrak{C}$. Hence by (3) $\mathfrak{Ig}(Kerf' \cap \mathfrak{C}) = \mathfrak{Ig}(Kerf \cap \mathfrak{C})$. So, $Kerf' = Kerf$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{i}(y) = f'(x)$. The map is well defined and is as required. ■

Proof. We now prove that (4) implies (5). Let $\mathfrak{C} \in L$. Let $\mathfrak{A}, \mathfrak{B} \in \mathbf{RCA}_\alpha$. Let $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorhisms. Then by the Neat Embedding Theorem, there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+ \in \mathbf{CA}_{\alpha+\omega}$ and embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{A}^+$ $e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{B}^+$ and $e_C : \mathfrak{C} \to \mathfrak{Nr}_\alpha \mathfrak{C}^+$. We can assume that $\mathfrak{S}g^{\mathfrak{A}^+} e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. Let $f(C)^+ = \mathfrak{S}g^{\mathfrak{A}^+} e_A(f(C))$ and $g(C)^+ = \mathfrak{S}g^{\mathfrak{B}^+} e_B(g(C))$. Then by (4) there exist $\bar{f} : \mathfrak{C}^+ \to f(C)^+$ and $\bar{g} : \mathfrak{C}^+ \to g(C)^+$ such that $(e_A \upharpoonright f(C)) \circ f = \bar{f} \circ e_C$ and $(e_B \upharpoonright g(C)) \circ g = \bar{g} \circ e_C$. Let $K = \{\mathfrak{A} \in \mathbf{CA}_{\alpha+\omega} : \mathfrak{A} = \mathfrak{S}g^{\mathfrak{A}} \mathfrak{Nr}_\alpha \mathfrak{A}\}$. Then $\mathfrak{A}^+$, $\mathfrak{B}^+$ and $\mathfrak{C}^+$ are all in K. Now by [3] 2.2.12 K has the amalgamation property, hence there is a $\mathfrak{D}^+$ in K and monomorphisms $k : \mathfrak{A}^+ \to \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \to \mathfrak{D}^+$ such that $k \circ \bar{f} = h \circ \bar{g}$. Let $\mathfrak{D} = \mathfrak{Nr}_\alpha \mathfrak{D}^+$. Then $k \circ e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{D}$ and $h \circ e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{D}$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. By this the proof is complete. ∎

Theorem

Theorem 2 Let $\alpha > 1$. Let $\beta = \alpha + \omega$. Then (1) - (5) in Lemma 1 are false for L $= \mathbf{RCA}_\alpha$.

Proof. Using Lemma 1 upon noting that $\mathbf{RCA}_\alpha$ fails to have the amalgamation property [3] and that $\mathbf{RCA}_\alpha$ satisfies the premise of (4) in Lemma 1 when $\beta = \alpha + \omega$. ∎

We readily conclude:

We cannot replace $\mathbf{Dc}_\alpha$ in 2.6.67 (ii), 2.6.71-72 of [1] by $\mathbf{RCA}_\alpha$ when $\alpha \geq \omega$. Lemma 1 tells us where to find direct counterexamples, namely from common subalgebras of algebras in $\mathbf{RCA}_\alpha$ that do not amalgamate.

All ordinals considered are infinite. Algebras of dimension α will be denoted by $\mathbf{FPA}_\alpha$. We show that every $\mathbf{FPA}_\alpha$ is representable, and this class has the superamalgamation property.

For $\alpha < \beta$, the neat α reduct of $\mathfrak{B} \in \mathbf{FPA}_\beta$ denoted by $\mathfrak{Nr}_\alpha\mathfrak{B}$ is defined like the polyadic case, except that cylindrifiers are finite. That is $\mathfrak{B} \in \mathbf{FPA}_\beta$ and $\alpha < \beta$ then $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$ is the $\mathbf{FPA}_\alpha$ whose domain is the set $\mathfrak{Nr}_\alpha B = \{b \in B : \Delta b \subseteq \alpha\}$, where $\Delta b = \{i \in \alpha : \mathsf{c}_i b \neq b\}$ with cylindrifiers restricted to α and for $\tau \in {}^\alpha\alpha$, and $a \in A$, $\mathsf{s}_\tau^{\mathfrak{A}} a = \mathsf{s}_{\bar{\tau}}^{\mathfrak{B}} a$ where $\bar{\tau} = \tau \cup Id_{\beta\sim\alpha}$.

Theorem

Let $\mathfrak{A} \in \mathbf{FPA}_\alpha$. *Then for all* $\beta \supseteq \alpha$, *there exists* $\mathfrak{B} \in \mathbf{FPA}_\beta$ *such that* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ *and for all* $X \subseteq A$ *one has* $\mathfrak{S}g^{\mathfrak{A}} X = \mathfrak{Nr}_\alpha \mathfrak{S}g^{\mathfrak{B}} X$.

Proof.

1 Let $(\mathfrak{A}, \alpha, S)$ be a transformation system. That is $\mathfrak{A}$ is a Boolean algebra and $S : ({}^{\alpha}\alpha, \circ) \to End(\mathfrak{A})$ is a homomorphism, where $End(\mathfrak{A})$ is the semigroup of all endomorphisms of $\mathfrak{A}$, with the operation of composition of maps. For any set X, let $F({}^{\alpha}X, \mathfrak{A})$ be the set of all functions from ${}^{\alpha}X$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in {}^{\alpha}\alpha$ and $f \in F({}^{\alpha}X, \mathfrak{A})$, $\mathsf{s}_{\tau}f(x) = f(x \circ \tau)$. This turns $F({}^{\alpha}X, \mathfrak{A})$ to a transformation system as well. The map $H : \mathfrak{A} \to F({}^{\alpha}\alpha, \mathfrak{A})$ defined by $H(p)(x) = \mathsf{s}_{x}p$ is easily checked to be an isomorphism. Assume that $\beta \supseteq \alpha$. Then $K : F({}^{\alpha}\alpha, \mathfrak{A}) \to F({}^{\beta}\alpha, \mathfrak{A})$ defined by $K(f)x = f(x \upharpoonright \alpha)$ is an isomorphism. These facts are straightforward to establish.

□

Proof.

1 $F({}^{\beta}\alpha, \mathfrak{A})$ is called a minimal dilation of $F({}^{\alpha}\alpha, \mathfrak{A})$. Elements of the big algebra, or the dilation, are of form $s_\sigma p$, $p \in F({}^{\beta}\alpha, \mathfrak{A})$ where σ is one to one on α. We say that $J \subseteq I$ supports an element $p \in A$ if whenever σ_1 and σ_2 are transformations that agree on J, then $s_{\sigma_1} p = s_{\sigma_2} p$. $\mathfrak{Nr}_J A$, consisting of the elements that J supports, is called a compression of $\mathfrak{A}$; with the operations defined the obvious way. If $\mathfrak{A}$ is an $\mathfrak{B}$ valued I transformation system wih domain X, then the J compression of $\mathfrak{A}$ is isomorphic to a $\mathfrak{B}$ valued J transformation system via $H : \mathfrak{Nr}_J\mathfrak{A} \to F({}^{J}X, \mathfrak{A})$ by setting for $f \in \mathfrak{Nr}_J\mathfrak{A}$ and $x \in {}^{J}X$, $H(f)x = f(y)$ where $y \in X^I$ and $y \upharpoonright J = x$.

□

Proof.

2 Now let $\alpha \subseteq \beta$. If $|\alpha| = |\beta|$ then the the required algebra is defined as follows. Let μ be a bijection from β onto α. For $\tau \in {}^{\beta}\beta$, let $\mathsf{s}_\tau = \mathsf{s}_{\mu\tau\mu^{-1}}$ and for each $i \in \beta$ let $\mathsf{c}_i = \mathsf{c}_{\mu(i)}$. Then this defined $\mathfrak{B} \in \mathit{FPA}_\beta$ in which $\mathfrak{A}$ neatly embeds via $\mathsf{s}_{\mu\restriction\alpha}$

$\square$

Proof.

3 Now assume that $|\alpha| < |\beta|$. Let $\mathfrak{A}$ be a given polyadic algebra of dimension α; discard its cylindrifiers and then take its minimal dilation $\mathfrak{B}$, which exists by the above. We need to define cylindrifiers on the big algebra, so that $\mathfrak{A} \cong \mathfrak{Nr}_\alpha\mathfrak{B}$. We let (*):

$$\mathsf{c}_k \mathsf{s}_\sigma^{\mathfrak{B}} p = \mathsf{s}_{\rho^{-1}}^{\mathfrak{B}} \mathsf{c}_{(\rho\{k\}\cap\sigma\alpha)} \mathsf{s}_{(\rho\sigma\restriction\alpha)}^{\mathfrak{A}} p.$$

Here ρ is a any permutation. It can be checked that this definition is sound; it is independent of the choice of ρ, and is as required. Furthermore, it defines the required algebra $\mathfrak{B}$. To prove the second part, abusing notation we write $\mathfrak{A}$ for $\mathfrak{S}g^{\mathfrak{A}}X$ and $\mathfrak{B}$ for $\mathfrak{S}g^{\mathfrak{B}}X$. Then $\mathfrak{B}$ is a minimal dilation of $\mathfrak{A}$. Each element of $\mathfrak{B}$ has the form $\mathsf{s}_\sigma^{\mathfrak{B}} a$ for some $a \in A$, and σ a transformation on β such that $\sigma \restriction \alpha$ is one to one. Then proceed as in the above lemma.

$\square$

For $\mathfrak{A} \in \mathbf{FPA}_\alpha$ a polyadic algebra and $\beta > \alpha$, a β dilation of $\mathfrak{A}$ is an algebra $\mathfrak{B} \in \mathbf{FPA}_\beta$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$. $\mathfrak{B}$ is a minimal dilation of $\mathfrak{A}$ if A generates $\mathfrak{B}$. Let $\mathbf{L} = \{\mathfrak{A} \in \mathbf{FPA}_\beta : \mathfrak{S}g\mathfrak{Nr}_\alpha A = A\}$. Then $\mathfrak{Nr}_\alpha : \mathbf{L} \to \mathbf{FPA}_\alpha$ is an equivalence. To prove this, we need a lemma. For $X \subseteq A$, $\mathfrak{Ig}^A X$ denotes the ideal generated by A:

Lemma

Let $\alpha < \beta$ be infinite ordinals. Let $\mathfrak{B} \in \mathbf{FPA}_\beta$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$.

1. *if A generates $\mathfrak{B}$ then $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$*
2. *If A generates $\mathfrak{B}$, and I is an ideal of $\mathfrak{B}$, then $\mathfrak{Ig}^{\mathfrak{B}}(I \cap A) = I$*

Proof.

1 Let $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ and A generates $\mathfrak{B}$ then $\mathfrak{B}$ consists of all elements $s_\sigma^{\mathfrak{B}} x$ such that $x \in A$ and σ is a transformation on β such that $\sigma \upharpoonright \alpha$ is one to one. Now suppose $x \in \mathfrak{Nr}_\alpha \mathfrak{S}g^{\mathfrak{B}} X$ and $\Delta x \subseteq \alpha$. There exists $y \in \mathfrak{S}g^{\mathfrak{A}} X$ and a transformation σ of β such that $\sigma \upharpoonright \alpha$ is one to one and $x = s_\sigma^{\mathfrak{B}}$. Let τ be a transformation of β such that $\tau \upharpoonright \alpha = Id$ and $(\tau \circ \sigma)\alpha \subseteq \alpha$. Then $x = s_\tau^{\mathfrak{B}} x = s_\tau^{\mathfrak{B}} s_\sigma y = s_{\tau \circ \sigma}^{\mathfrak{B}} y = s_{\tau \circ \sigma \upharpoonright \alpha}^{\mathfrak{A}'} y$.

$\square$

Proof.

2 Only one inclusion is non-trivial. Let $x \in \mathfrak{Ig}^{\mathfrak{B}}(I \cap A)$. Then $\mathsf{c}_{(\Delta x \sim \alpha)} x \in \mathfrak{Nr}_\alpha \mathfrak{B} = \mathfrak{A}$, hence in $I \cap A$. But $x \leq \mathsf{c}_{(\Delta x \sim \alpha)} x$, and we are done.

$\square$

The previous lemma fails for cylindric algebras, but it does hold for the class $\mathbf{Dc}_\alpha$'s of so-called dimension complemented algebras.

Theorem

*Let $\alpha < \beta$ be infinite ordinals. Assume that $\mathfrak{A}, \mathfrak{A}' \in$ **FPA**$_\alpha$ and $\mathfrak{B}, \mathfrak{B}' \in$ **FPA**$_\beta$. If $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}'$ and A generates both then $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic, then $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic with an isomorphism that fixes $\mathfrak{A}$ pointwise.*

Proof.

1 We prove something stronger, we assume that $\mathfrak{A}$ embeds into $\mathfrak{Nr}_\alpha\mathfrak{B}$ and similarly for $\mathfrak{A}'$. So let $\mathfrak{A}, \mathfrak{A}' \in \mathbf{FPA}_\alpha$ and $\beta > \alpha$. Let $\mathfrak{B}, \mathfrak{B}' \in \mathbf{FPA}_\beta$ and assume that $e_A, e_{A'}$ are embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{Nr}_\alpha\mathfrak{B}, \mathfrak{Nr}_\alpha\mathfrak{B}'$, respectively, such that $\mathfrak{S}g^{\mathfrak{B}}(e_A(A)) = \mathfrak{B}$ and $\mathfrak{S}g^{\mathfrak{B}'}(e_{A'}(A')) = \mathfrak{B}'$, and let $i : \mathfrak{A} \longrightarrow \mathfrak{A}'$ be an isomorphism. We need to "lift" i to β dimensions.

$\square$

Proof.

2 Let $\mu = |A|$. Let x be a bijection from μ onto A. Let y be a bijection from μ onto A', such that $i(x_j) = y_j$ for all $j < \mu$. Let $\mathfrak{D} = \mathfrak{Fr}_\mu \mathbf{FPA}_\beta$ with generators $(\xi_i : i < \mu)$. Let $\mathfrak{C} = \mathfrak{S}g^{\mathfrak{Rd}_\alpha \mathfrak{D}}\{\xi_i : i < \mu\}$. Then $\mathfrak{C} \subseteq \mathfrak{Nr}_\alpha \mathfrak{D}$, C generates $\mathfrak{D}$ and so by the previous lemma $\mathfrak{C} = \mathfrak{Nr}_\alpha \mathfrak{D}$. There exist $f \in Hom(\mathfrak{D}, \mathfrak{B})$ and $f' \in Hom(\mathfrak{D}, \mathfrak{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and $f'(g_\xi) = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that f and f' are both onto. We now have $e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \upharpoonright \mathfrak{C}) = f \upharpoonright \mathfrak{C}$. Therefore $Kerf' \cap \mathfrak{C} = Kerf \cap \mathfrak{C}$. Hence by $\mathfrak{Ig}(Kerf' \cap \mathfrak{C}) = \mathfrak{Ig}(Kerf \cap \mathfrak{C})$. So, again by the the previous lemma, $Kerf' = Kerf$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{i}(y) = f'(x)$. The map is well defined and is as required.

$$
\begin{tikzcd}
D \arrow[r, "f"] \arrow[dr, "f'"'] & B \arrow[d, "\hat{i}"] & A \arrow[l, "e_A"'] \arrow[d, "i"] \\
 & B' & \mathcal{A}' \arrow[l, "e_{A'}"]
\end{tikzcd}
$$

Corollary

Let $\mathfrak{A}, \mathfrak{A}', i, e_A, e_{A'}, \mathfrak{B}$ and $\mathfrak{B}'$ be as in the previous proof. Then if i is a monomorphism form $\mathfrak{A}$ to $\mathfrak{A}'$, then it lifts to a monomorphism $\bar{i}$ from $\mathfrak{B}$ to $\mathfrak{B}'$.

Proof.

Consider $i : \mathfrak{A} \to i(\mathfrak{A})$. Take $\mathfrak{C} = \mathfrak{S}g^{\mathfrak{B}'}(e_{A'}i(A))$. Then i lifts to an isomorphism $\bar{i} \to \mathfrak{C} \subseteq \mathfrak{B}$.

$$\begin{tikzcd}
B \arrow[d, "\hat{i}"] & A \arrow[l, "e_A"'] \arrow[d, "i"] \\
B' & \mathcal{A}' \arrow[l, "e_{A'}"]
\end{tikzcd}$$

$\square$

We assume familiarity with basic concepts in category theory like functors, natural transformations, adjoint situations.

Theorem

Let $\beta > \alpha$. *Let* $\mathbf{L} = \{\mathfrak{A} \in \mathbf{FPA}_\beta : \mathfrak{A} = \mathfrak{S}g^{\mathfrak{A}}\mathfrak{Nr}_\alpha\mathfrak{A}\}$. *Let* $\mathfrak{Nr} : \mathbf{L} \to \mathbf{FPA}_\alpha$ *be the neat reduct functor. Then* $\mathfrak{Nr}$ *is invertible. That is, there is a functor* $G : \mathbf{FPA}_\alpha \to \mathbf{L}$ *and natural isomorphisms* $\mu : 1_{\mathbf{L}} \to G \circ \mathfrak{Nr}$ *and* $\epsilon : \mathfrak{Nr} \circ G \to 1_{\mathbf{FPA}_\alpha}$.

Proof.

The idea is that a full, faithful, dense functor is invertible. Let L be a system of representatives for isomorphism on $Ob(\mathbf{L})$. For each $\mathfrak{B} \in Ob(\mathbf{FPA}_\alpha)$ there is a unique $\mathfrak{G}(B)$ in L such that $\mathfrak{Nr}(G(\mathfrak{B})) \cong \mathfrak{B}$. $G(\mathfrak{B})$ is a minmal dilation of $\mathfrak{B}$. Then $G : Ob(\mathbf{FPA}_\alpha) \to Ob(\mathbf{L})$ is well defined. Choose one isomorphism $\epsilon_B : \mathfrak{Nr}(G(B)) \to \mathfrak{B}$. If $g : \mathfrak{B} \to \mathfrak{B}'$ is a $\mathbf{FPA}_\alpha$ morphism, then the square

$$\begin{tikzcd}
\mathfrak{Nr}(G(B)) \arrow[r, "\epsilon_B"] \arrow[d, "\epsilon_B^{-1}\circ g\circ\epsilon_{B'}"'] & B \arrow[d, "g"] \\
\mathfrak{Nr}(G(B')) \arrow[r, "\epsilon_{B'}"'] & B'
\end{tikzcd}$$

commutes. □

Proof.

By corollary 0.1, there is a unique morphism $f : G(\mathfrak{B}) \to G(\mathfrak{B}')$ such that $\mathfrak{Nr}(f) = \epsilon_{\mathfrak{B}}^{-1} \circ g \circ \epsilon$. We let $G(g) = f$. Then it is easy to see that G defines a functor. Also, by definition $\epsilon = (\epsilon_{\mathfrak{B}})$ is a natural isomorphism from $\mathfrak{Nr} \circ G$ to $1_{\mathbf{FPA}_\alpha}$. To find a natural isomorphism from $1_{\mathbf{L}}$ to $G \circ \mathfrak{Nr}$, observe that $e_{FA} : \mathfrak{Nr} \circ G \circ \mathfrak{Nr}(\mathfrak{A}) \to \mathfrak{Nr}(\mathfrak{A})$ is an isomorphism. Then there is a unique $\mu_A : \mathfrak{A} \to G \circ \mathfrak{Nr}(\mathfrak{A})$ such that $\mathfrak{Nr}(\mu_{\mathfrak{A}}) = e_{FA}^{-1}$. □

Proof.

Since ϵ^{-1} is natural for any $f : \mathfrak{A} \to \mathfrak{A}'$ the square

$$\begin{tikzcd}
\mathfrak{Nr}(A) \arrow[r, "\epsilon^{-1}_{\mathfrak{Nr}(A)}=\mathfrak{Nr}(\mu_A)"] \arrow[d, "\mathfrak{Nr}(f)"'] & \mathfrak{Nr} \circ G \circ \mathfrak{Nr}(A) \arrow[d, "\mathfrak{Nr}\circ G\circ\mathfrak{Nr}(f)"] \\
\mathfrak{Nr}(A') \arrow[r, "\epsilon^{-1}_{FA}=\mathfrak{Nr}(\mu_{A'})"'] & \mathfrak{Nr} \circ G \circ \mathfrak{Nr}(A')
\end{tikzcd}$$

commutes, hence the square

$$\begin{tikzcd}
A \arrow[r, "\mu_A"] \arrow[d, "f"'] & G \circ \mathfrak{Nr}(A) \arrow[d, "G\circ\mathfrak{Nr}(f)"] \\
A' \arrow[r, "\mu_{A'}"'] & G \circ \mathfrak{Nr}(A')
\end{tikzcd}$$

commutes, too. Therefore $\mu = (\mu_A)$ is as required. $\square$

Theorem

Let β be a cardinal, and $\mathfrak{A} = \mathfrak{Fr}_\beta \mathbf{FPA}_\alpha$ be the free algebra on β generators. Let $X_1, X_2 \subseteq \beta$, $a \in \mathfrak{Sg}^{\mathfrak{A}} X_1$ and $c \in \mathfrak{Sg}^{\mathfrak{A}} X_2$ be such that $a \leq c$. Then there exists $b \in \mathfrak{Sg}^{\mathfrak{A}}(X_1 \cap X_2)$ such that $a \leq b \leq c$.

Proof.

Let $a \in \mathfrak{S}g^{\mathfrak{A}}X_1$ and $c \in \mathfrak{S}g^{\mathfrak{A}}X_2$ be such that $a \leq c$. We want to find an interpolant in $\mathfrak{S}g^{\mathfrak{A}}(X_1 \cap X_2)$. Assume that κ is a regular cardinal $> max(|\alpha|, |A|)$. Let $\mathfrak{B} \in \mathbf{FPA}_\kappa$ such that $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$, and A generates $\mathfrak{B}$. If an interpolant exists in $\mathfrak{B}$, then an interpolant exists in $\mathfrak{A}$. For assume that $a \leq b \leq c$, where $c \in Sg^{\mathfrak{B}}(X_1 \cap X_2)$. Since $\mathfrak{A}$ generates $\mathfrak{B}$ we have $|\Delta x \sim \alpha| < \omega$ for every $x \in B$. This can be proved by a simple inductive argument, with the base of the induction being the elements of $\mathfrak{A}$. Then there exists a finite $\Gamma \subseteq \kappa \sim \alpha$ such that $a \leq \mathsf{c}_{(\Gamma)}b \leq c$ and

$$\mathsf{c}_{(\Gamma)}b \in \mathfrak{Nr}_\alpha\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2) = \mathfrak{S}g^{\mathfrak{Nr}_\alpha\mathfrak{B}}(X_1 \cap X_2) = \mathfrak{S}g^{\mathfrak{A}}(X_1 \cap X_2).$$

Assume seeking a contradiction that no such interpolant exists in $\mathfrak{B}$. □

Proof.

1 Arrange $\kappa \times \mathfrak{S}g^{\mathfrak{C}}(X_1)$ and $\kappa \times \mathfrak{S}g^{\mathfrak{C}}(X_2)$ into κ-termed sequences:

$$\langle (k_i, x_i) : i \in \kappa \rangle \text{ and } \langle (l_i, y_i) : i \in \kappa \rangle \text{ respectively.}$$

Since κ is regular, we can define by recursion κ-termed sequences:

$$\langle u_i : i \in \kappa \rangle \text{ and } \langle v_i : i \in \kappa \rangle$$

such that for all $i \in \kappa$ we have:

$$u_i \in \kappa \smallsetminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}$$

and

$$v_i \in \kappa \smallsetminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}.$$

$\square$

Proof.

1 For a boolean algebra $\mathfrak{C}$ and $Y \subseteq \mathfrak{C}$, we write $fl^{\mathfrak{C}}Y$ to denote the boolean filter generated by Y in $\mathfrak{C}$. Now let

$$Y_1 = \{a\} \cup \{-\mathsf{c}_{k_i}x_i + \mathsf{s}_{u_i}^{k_i}x_i : i \in \omega\},$$

$$Y_2 = \{-c\} \cup \{-\mathsf{c}_{l_i}y_i + \mathsf{s}_{v_i}^{l_i}y_i : i \in \omega\},$$

$$H_1 = fl^{Bl\mathfrak{S}g^{\mathfrak{B}}(X_1)}Y_1,\ H_2 = fl^{Bl\mathfrak{S}g^{\mathfrak{B}}(X_2)}Y_2,$$

and

$$H = fl^{Bl\mathfrak{S}g^{\mathfrak{B}}(X_1\cap X_2)}[(H_1 \cap \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2) \cup (H_2 \cap \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)].$$

Then H is a proper filter of $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$. This is proved by induction with the base of the induction bieng no interpolant exists in $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ Let H^* be a (proper boolean) ultrafilter of $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ containing H.

□

Proof.

2 We obtain ultrafilters F_1 and F_2 of $\mathfrak{S}g^{\mathfrak{B}}X_1$ and $\mathfrak{S}g^{\mathfrak{B}}X_2$, respectively, such that

$$H^* \subseteq F_1, \ \ H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2) = H^* = F_2 \cap \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2).$$

Now for all $x \in \mathfrak{S}g^{\mathcal{B}}(X_1 \cap X_2)$ we have

$$x \in F_1 \text{ if and only if } x \in F_2.$$

Also from how we defined our ultrafilters, F_i for $i \in \{1, 2\}$ satisfy the following condition: (*) For all $k < \mu$, for all $x \in \mathfrak{S}g^{\mathfrak{B}}X_i$ if $\mathsf{c}_k x \in F_i$ then $\mathsf{s}_l^k x$ is in F_i for some $l \notin \Delta x$.

$\square$

Proof.

2 Let $\mathfrak{D}_i = \mathfrak{S}g^{\mathfrak{A}} X_i$. For a transformation $\tau \in {}^{\alpha}\kappa$ let $\bar{\tau} = \tau \cup Id_{\kappa \sim \alpha}$. Define f_i from $\mathfrak{D}_i$ to the full set algebra $\mathfrak{C}$ with unit ${}^{\alpha}\kappa$ as follows:

$$f_i(x) = \{\tau \in {}^{\alpha}\kappa : \mathsf{s}_{\bar{\tau}} x \in F_i\}, \text{ for } x \in \mathcal{D}_i$$

Then f_i is a homomorphism by (*). Without loss of generality, we can assume that $X_1 \cup X_2 = X$. By (**) we have f_1 and f_2 agree on $X_1 \cap X_2$. So that $f_1 \cup f_2$ defines a function on $X_1 \cup X_2$, by freeness it follows that there is a homomorphism f from $\mathfrak{B}$ to $\mathfrak{C}$ such that $f_1 \cup f_2 \subseteq f$. Then $q \in f(a) \cap f(-c) = f(a - c)$. This is so because $s_{Id} a = a \in F_1$ $s_{Id}(-c) = -c \in F_2$. But this contradicts the premise that $a \leq c$.

$\square$

Corollary

1. *Every* **FPA**$_\alpha$ *is representable as a subdirect product of set algebras.*
2. **FPA**$_\alpha$ *has the superamalgamation property*

Concluding Remarks

Call a system of varieties neat if it is a system of varieties definable by schemes satisfying the the finiteness generating condition and satisfying that for all $\mathfrak{A} \in K_\alpha$ there exists $\mathfrak{B} \in K_{\alpha+\omega}$ such that for all $X \subseteq A$, $\mathfrak{S}g^{\mathfrak{A}}X = \mathfrak{Nr}_\alpha \mathfrak{S}g^{\mathfrak{B}}X$. Call a system of varieties nice if the neat reduct functor has a right adjoint, and $K_\alpha = Kn_\alpha = S\mathfrak{Nr}_\alpha K_{\alpha+\omega}$ have *SUPAP*, and each K_α is axiomatized by a finite schema. Is there a neat or /and nice system of varieties definable by (finitely many) schemas? (In which case we only rquire that K_ω is definable by schemes.) This is a difficult question that lies at the heart of the process of algebriasation, and is strongly related to the so called finitizability problem in algebraic logic.

Concluding Remarks

A finitely axiomatizable variety V which shares the positive properties of $\mathbf{FPA}_\alpha$ and $\mathbf{Dc}_\alpha$ is that studied by Sain which shows that V has *SUPAP*. This is a situation where the positive properties of both paradigms amalgamate. Dropping the condition of definable by schemes, and modifying the definitions in the obvious way, such algebras can be viewed as a system which is both neat and nice. The system of varieties ($\mathbf{FPA}_\alpha : \alpha \geq \omega$) is also nice and neat.

Concluding Remarks

The neat reduct functor applied to representable cylindric algebras and some of its closer relatives like quasipolyadic algebras do not even have a right adjoint; and as it happens such classes of representable algebras do not have the amalgamation property. tuitive level the equiavlence defined above shows that for algebras in any infinite dimension, we have infinitely many spare dimensions , and the existence of the inverse functor tells us that this really does not take us out from the former category, since terms definable in these extra dimensions, are already term definable. This is another way of expressing definability properties like those of Beth and Craig. For $\mathbf{Dc}_\alpha$ we stipulate fom the very start that these spare dimensions exist, we get many positive results, but we pay the price that the class is not a variety.

Concluding Remarks

In $\mathbf{FPA}_\alpha$, and for that matter full polyadic algebras, and their countable reducts studie by Sain we can actually create extra dimensions, expressed algebraically by dilations and implemented via deep neat embedding theorems, which are basically Henkin constructions in algebraic disguise.

Concluding Remarks

Another dichotomy betwen the polyadic paradigm and the cylindric one is that in the former case the neat reduct functor is an equivalence, while in the latter it does not even have a right adjoint.

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