Tomasz Placek

Relativity and modal logic meet Hausdorff

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honoring István Németi's 70th birthday
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Aim:
to construct structures that generalize GR spacetimes and allow for indeterminism

The Hausdorff property

Let $\mathcal{T}(X)$ be a topology on set X. $\mathcal{T}(X)$ has the Hausdorff property iff for every $x, y \in X$ there are $O_x, O_y \in \mathcal{T}(X)$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$

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Logic for (in)determinism, tenses, and agency (Prior-Kripke-Thomason, Belnap, ...)

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Interplay of tenses, possibilities and agents' actions

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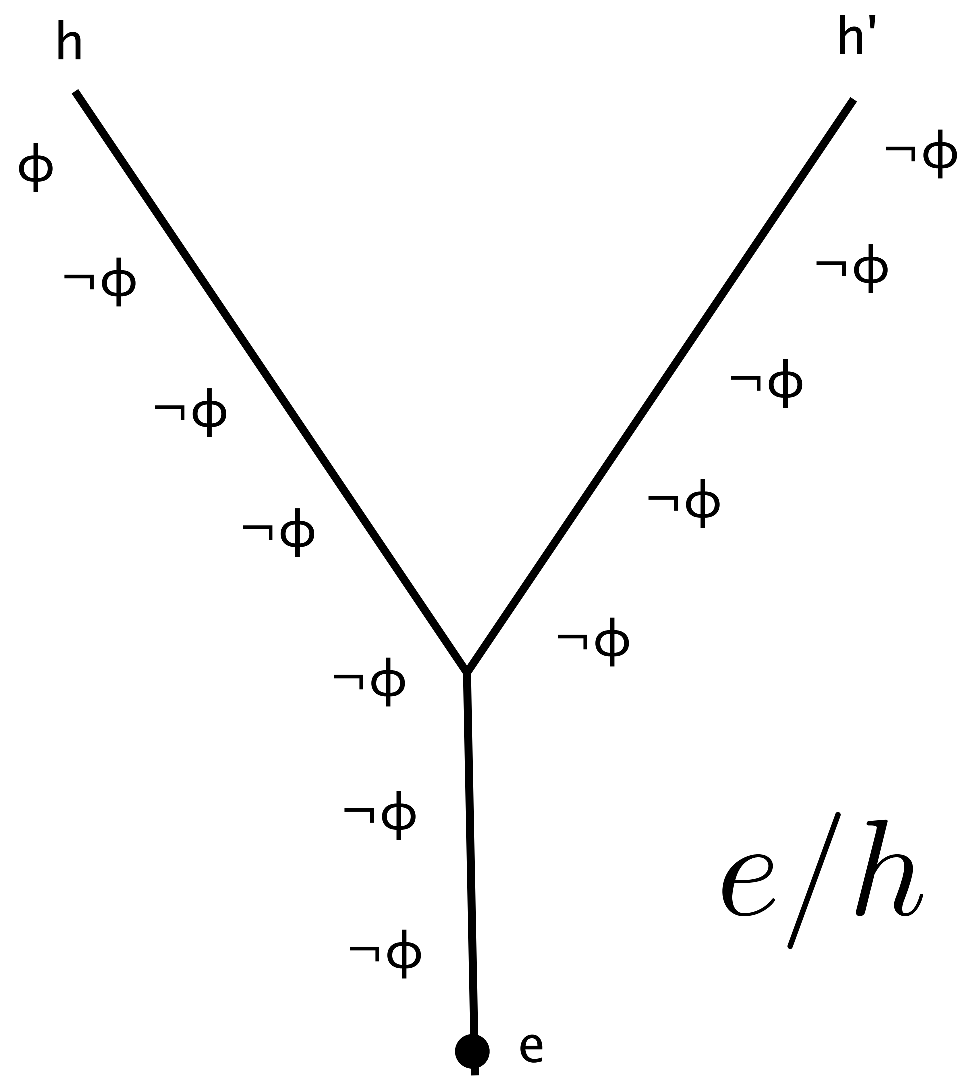
Interplay of tenses, possibilities and agents' actions

Yesterday both X and non-X were possible.
Today it is already settled that X occurred.

Branching Time (BT) semantics (Kripke -Prior-Thomason): formulas are evaluated at the event/history pairs, i.e., $\mathfrak{M}, e/h \models \varphi$



$e/h \models F : \phi$ but $e/h' \models \neg F : \phi$

Problem: hyper-events

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- simultaneity assumed, BT is a non-relativistic framework

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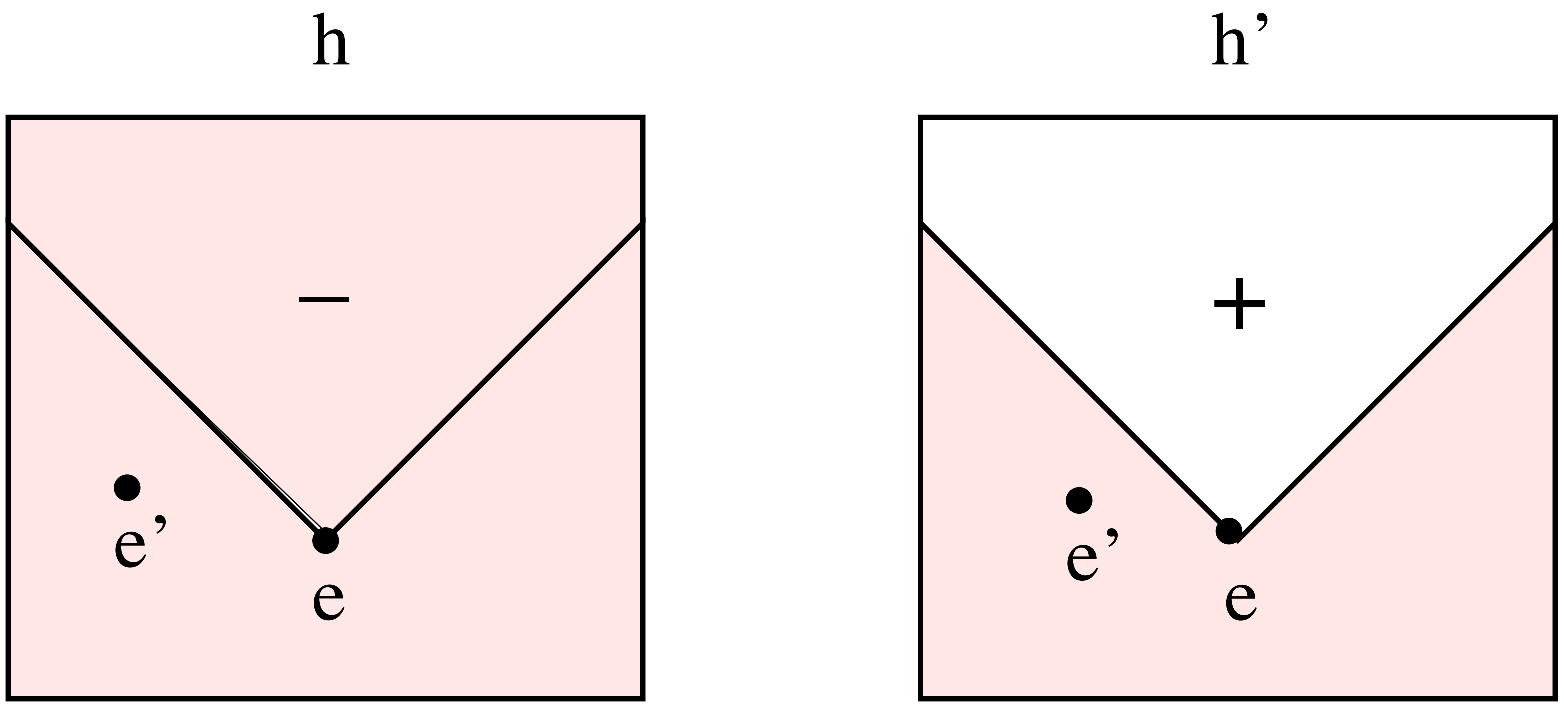
- simultaneity assumed, BT is a non-relativistic framework

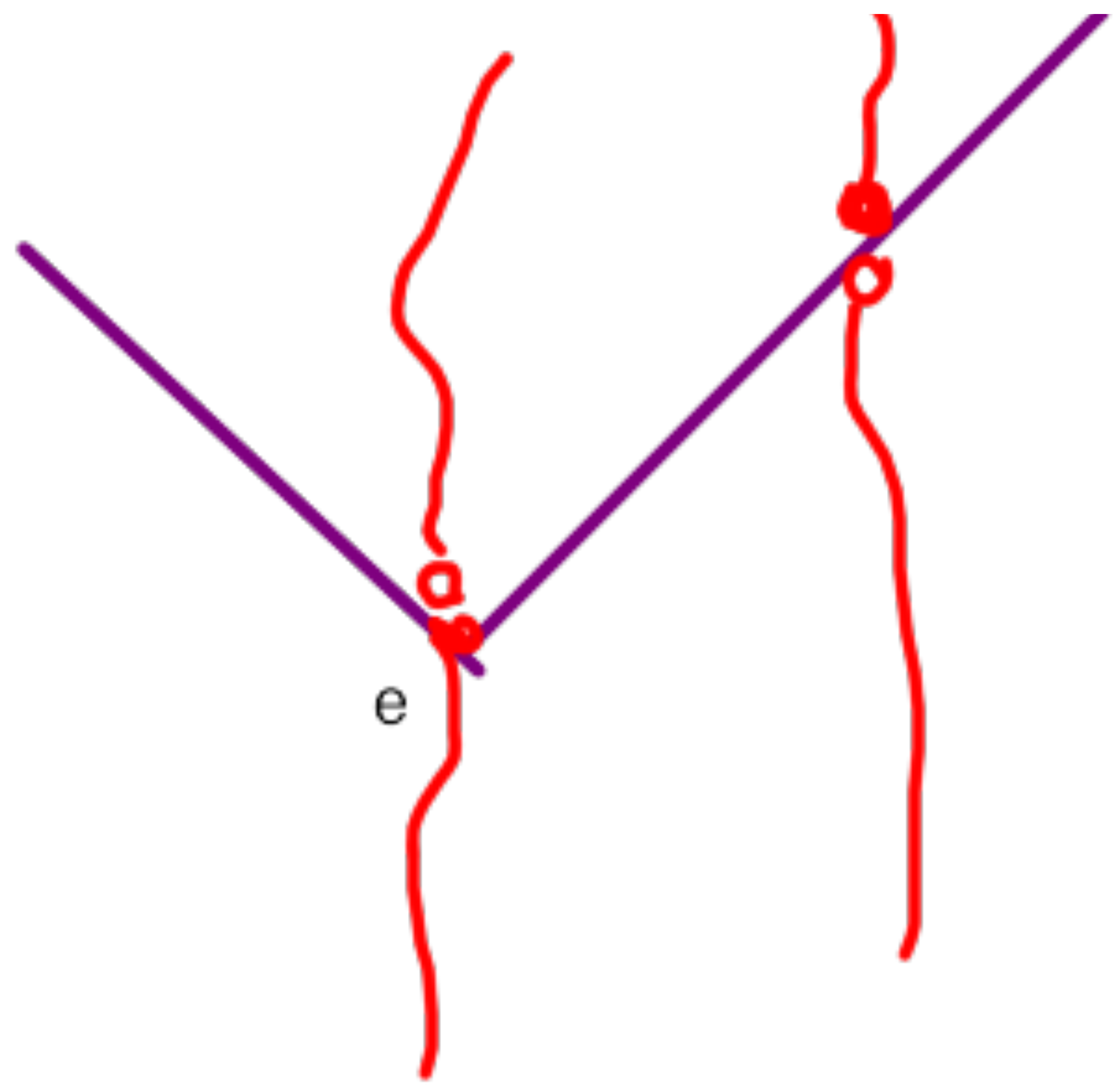
- the Pittsburgh worry: no room for actions of independent agents

Solution:
Branching Space-Times (BST) (Belnap 1992)

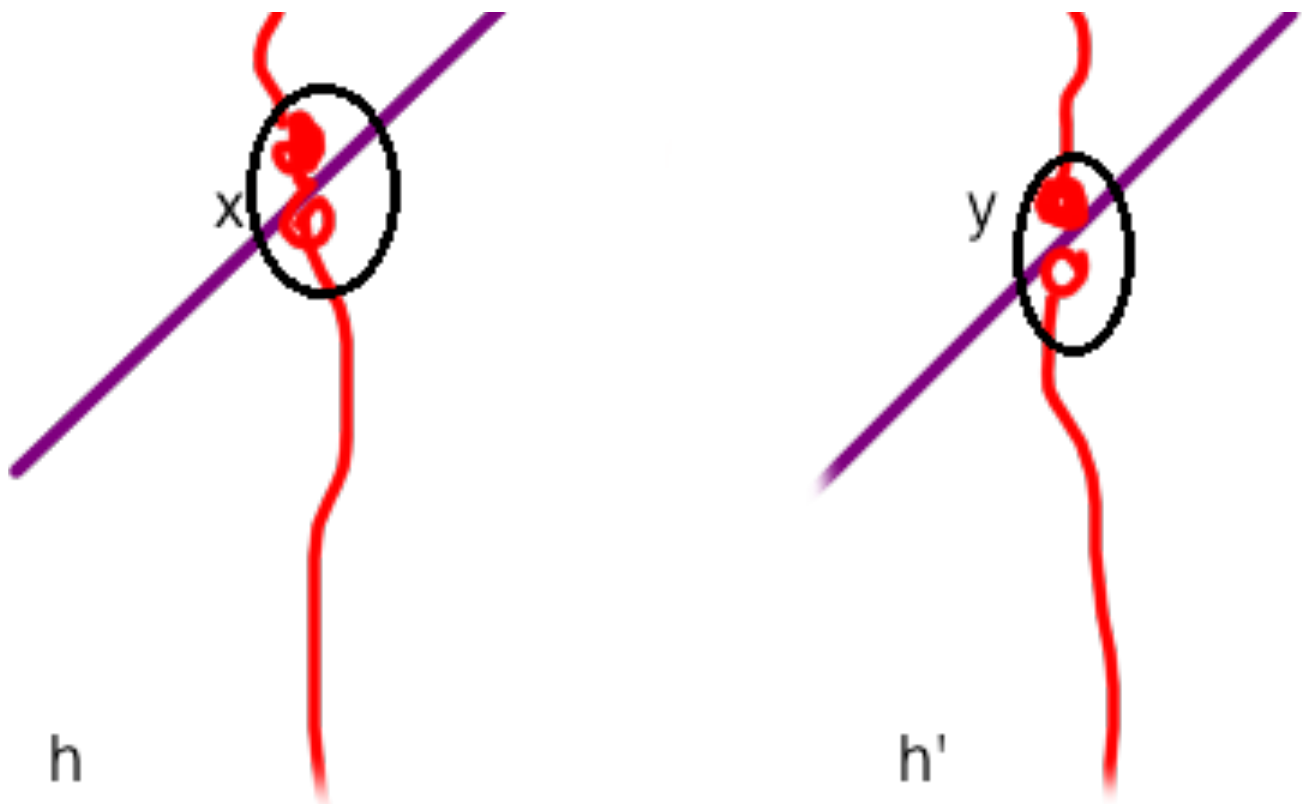


Two histories (h, h’), one choice point (e), i.e., a maximal point in the intersection of h and h’.

Different behavior of maximal chains in a history



Failure of the Hausdorff property ?



Results (Belnap, Kishida, Placek):
a multi-history BST model is generically non-Hausdorff (in the Bartha topology);

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Direction: a spacetime/history is Hausdorff, a bundle of spacetimes is not Hausdorff

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A reaction to singularity theorems (Hawking, Penrose, early 1970's) -
allow for non-Hausdorff spacetimes

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A series of blows:
non-Hausdorff spacetime violates the strong causality condition (Hajicek 1972).

Non-Hausdorff comes with a price (for a survey, see Earman 2008)

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Back to sanity:

"I must ... return firmly to sanity by repeating to myself three times: spacetime is a Hausdorff differentiable manifold; spacetime is a Hausdorff ..."(Penrose 1979).

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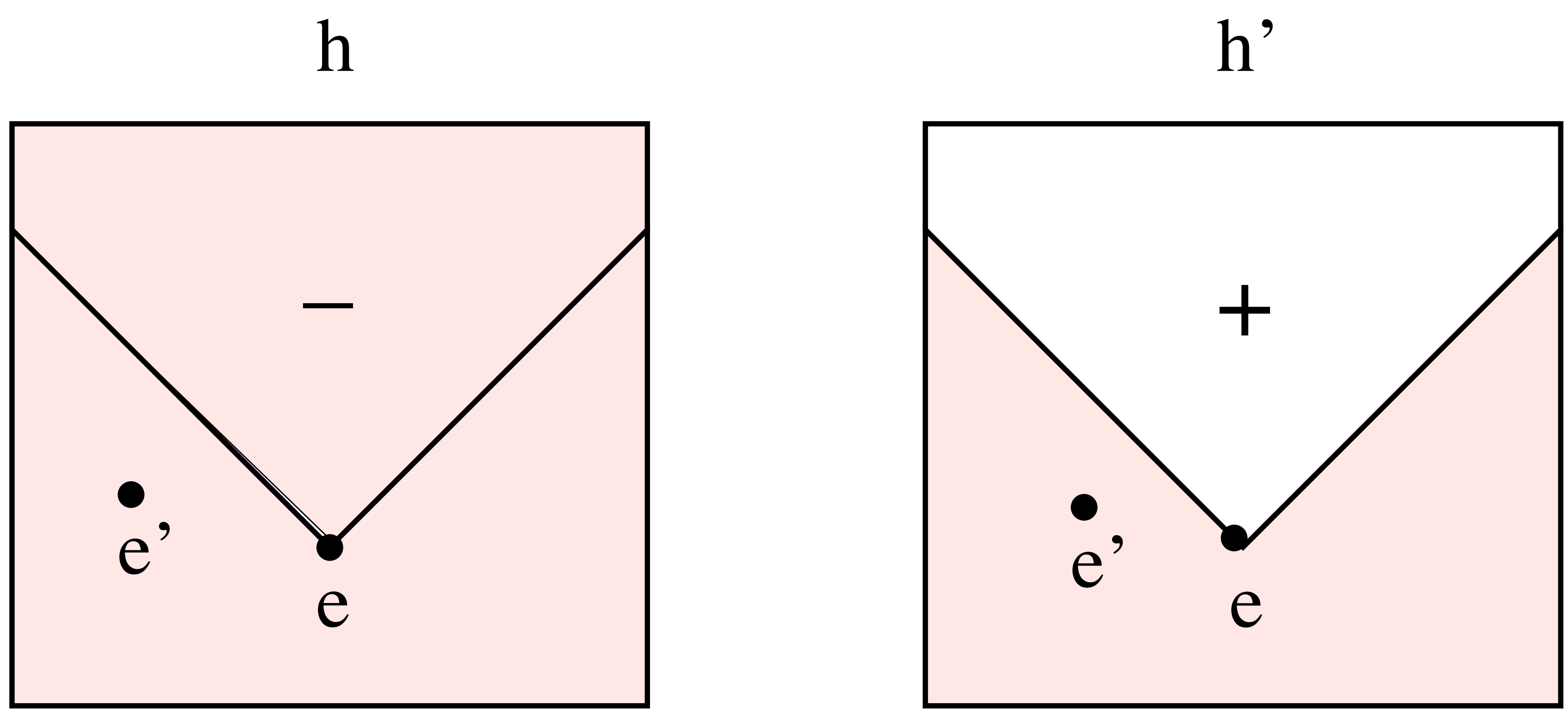

Project:

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Begin with a larger structure which has room for spacetime and for alternative possibilities.

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Begin with a larger structure which has room for spacetime and for alternative possibilities. Define spacetimes as maximal Hausdorff submanifolds

Idea: take the concept of normal convex set from GR; since BST comes with the notion of alternative possibilities, generalize it to make room for alternative possibilitires. Call it: patch.

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A pair $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$, where $W \neq \emptyset$, $\preceq$ is a pre-order on W, and $\mathcal{O} \subseteq \mathcal{P}(W)$, is a generalized BST model iff for every $e \in W$ there is a set of patches $\mathcal{O}_e \subseteq \mathcal{O}$ around e such that for all $O \in \mathcal{O}_e$:

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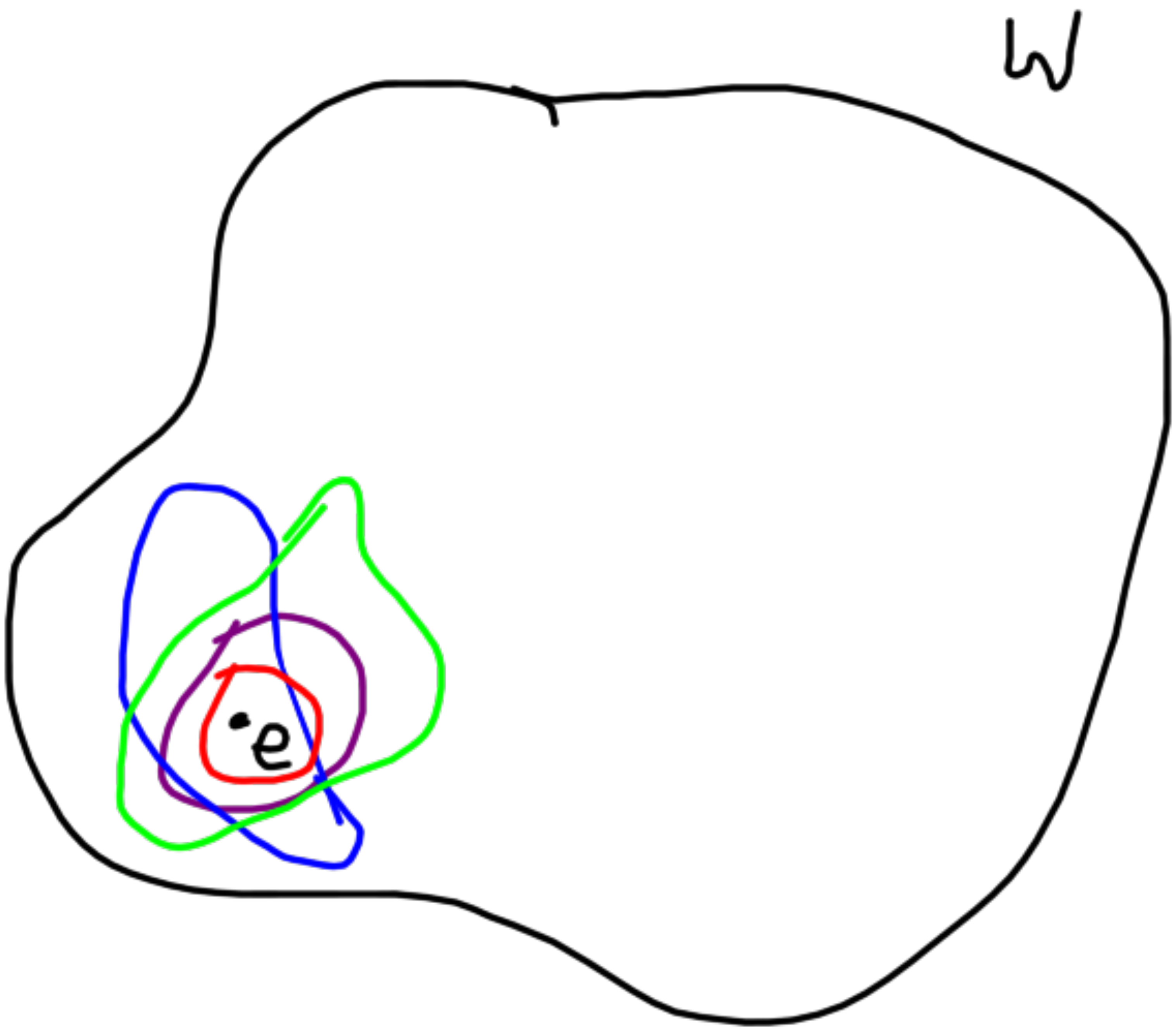

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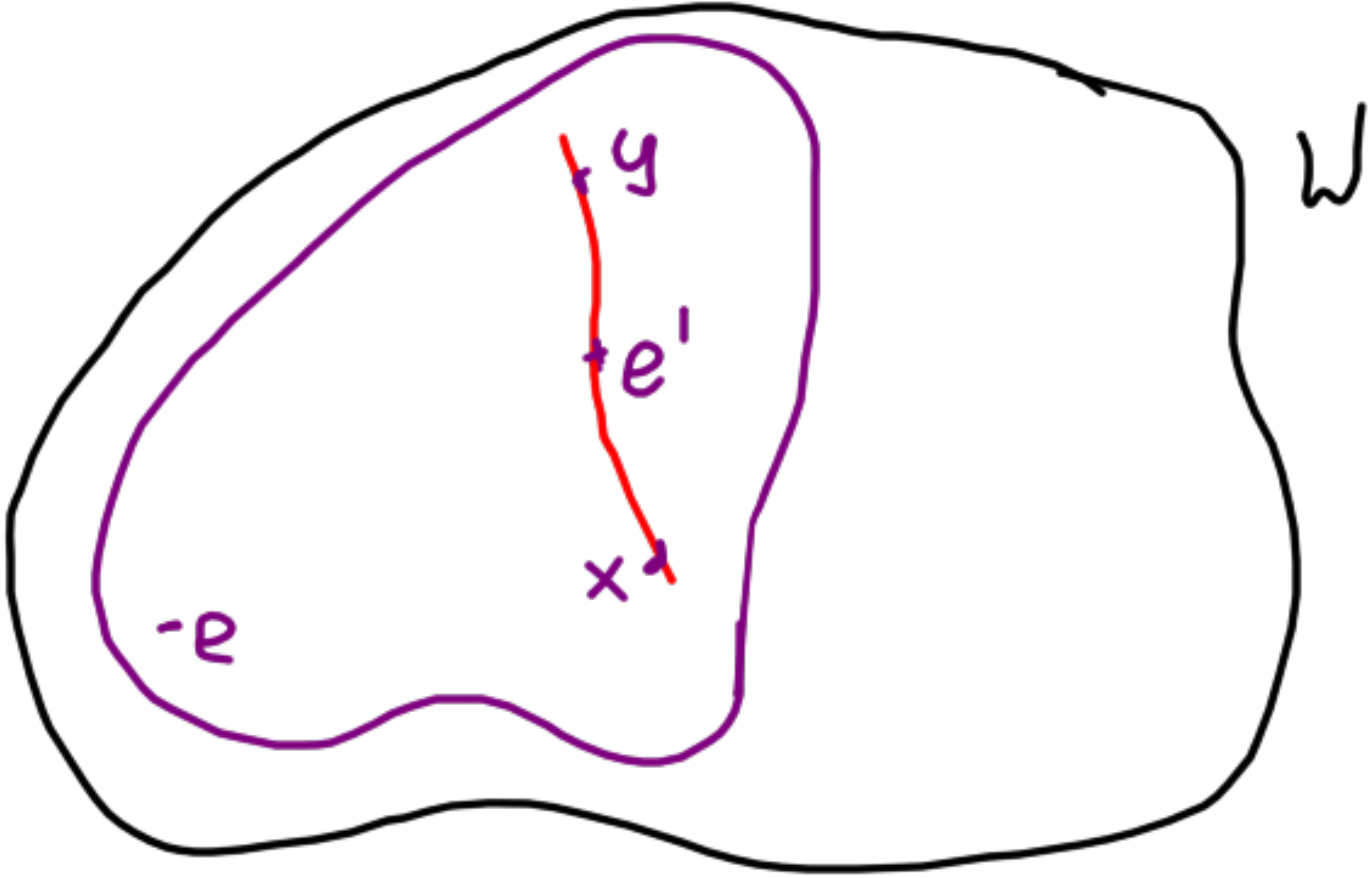

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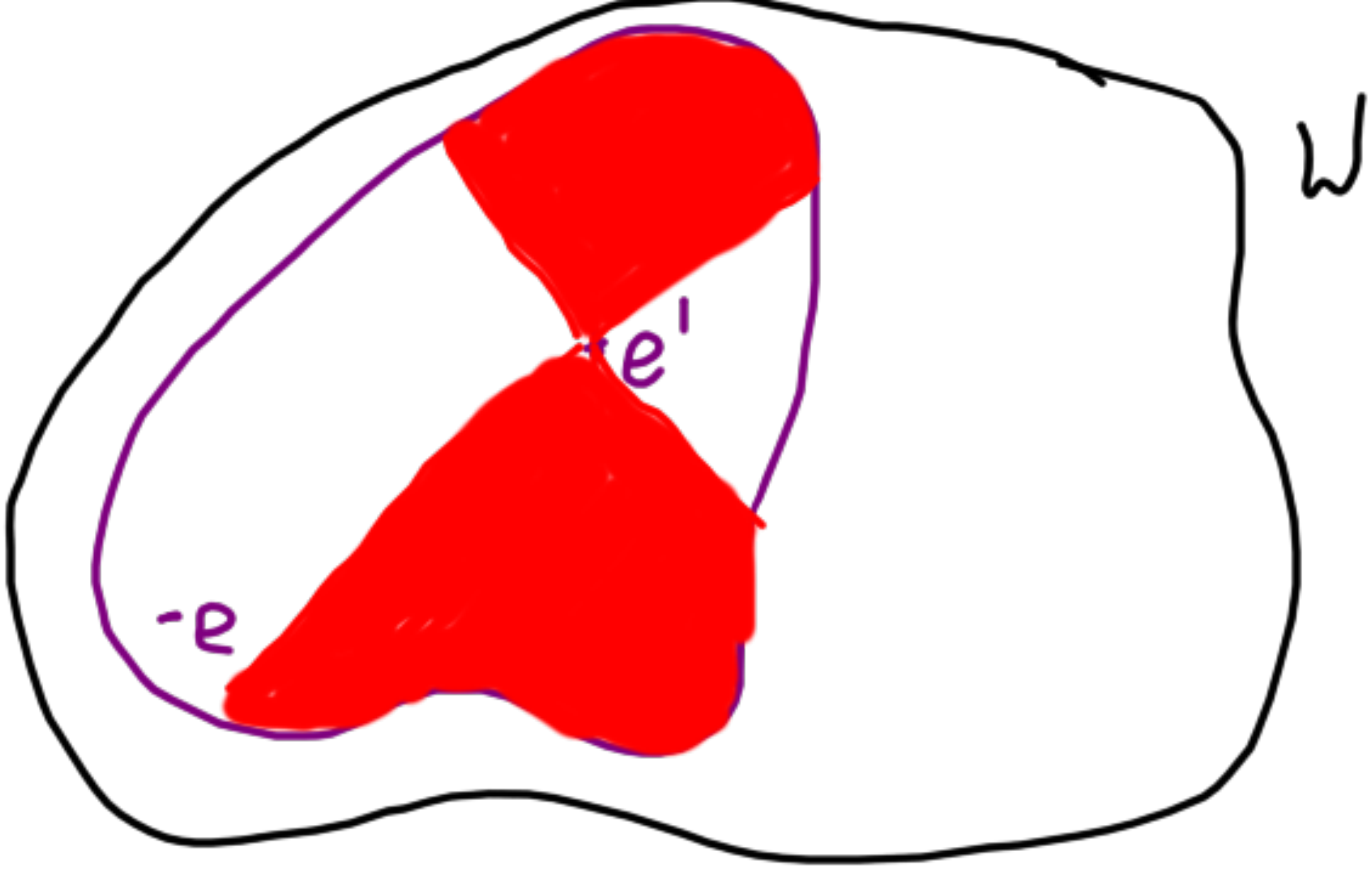

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– every lower bounded chain in $\langle O, \preceq_{|O}\rangle$ has an infimum in O;

– if a chain C in $\langle O, \preceq_{|O} \rangle$ is upper bounded by $b \in O$, then $B := \{x \in O_e \mid C \preceq_{|O} x \wedge x \preceq_{|O} b\}$ has a unique minimum,

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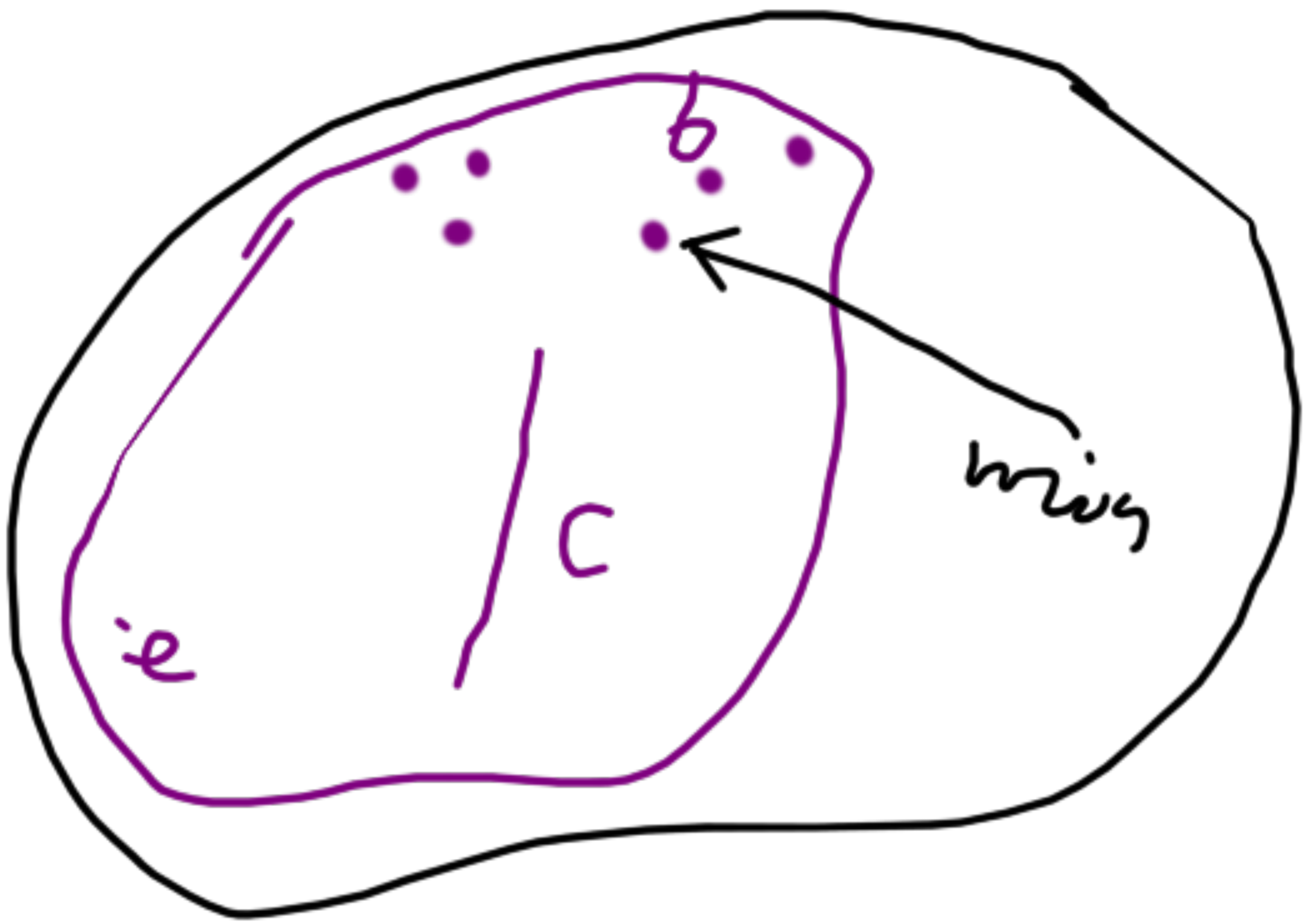

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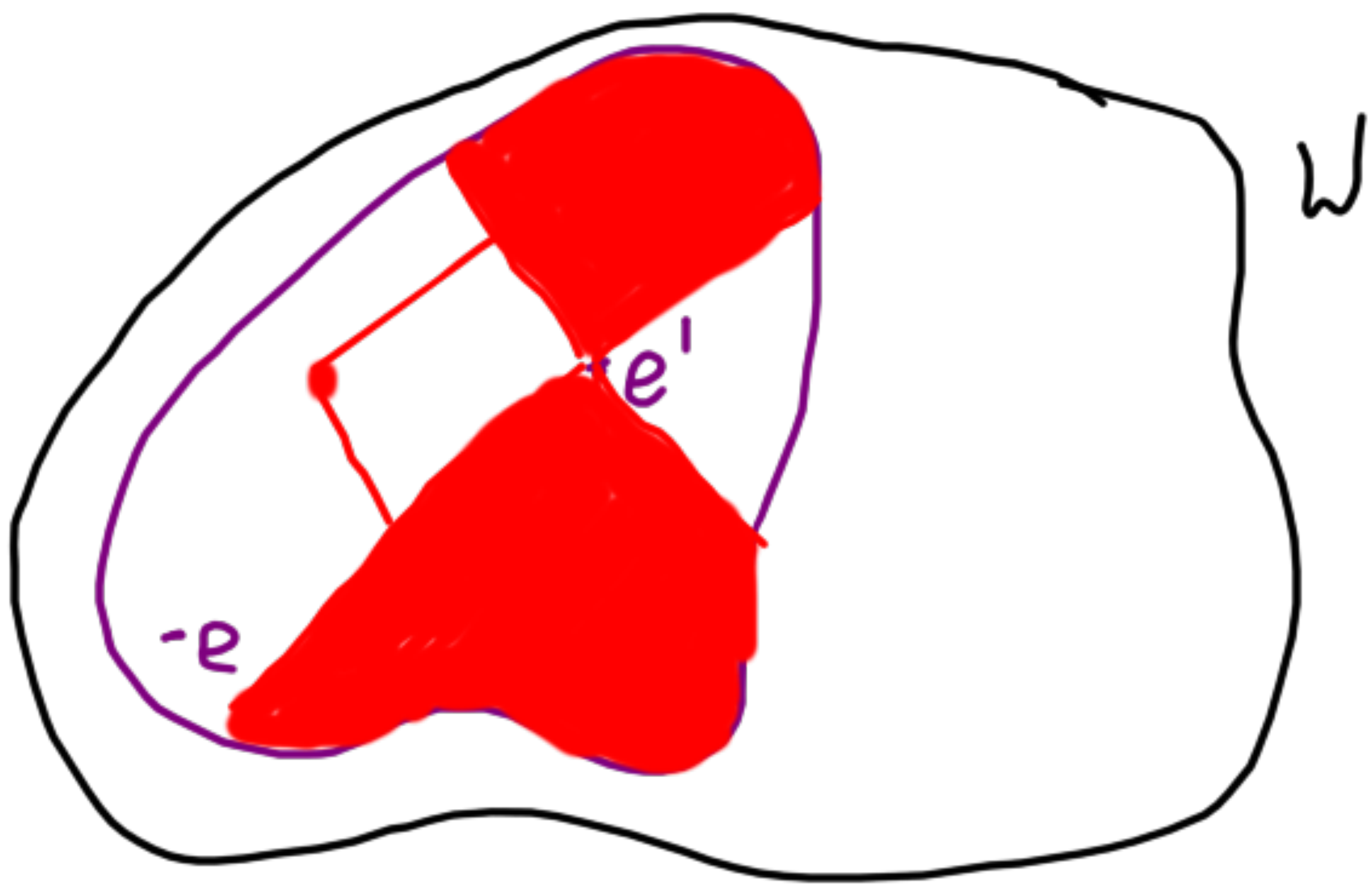

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3. $\mathcal{O} = \bigcup\{\mathcal{O}_e \mid e \in W\}$;

4. If $x, y \in O \cap O'$, where $O, O' \in \mathcal{O}$, then $x \preceq_{|O} y$ iff $x \preceq_{|O'} y$.

Seeds of modal inconsistency: splitting pairs

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Let $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$ be a generalized BST model and $O \in \mathcal{O}$. We say that $e, e' \in O$ form a splitting pair in O, iff $e \neq e'$ and there is a chain C in $\langle O, \preceq_{|O} \rangle$ such that e and e' are minimal upper bounds of C in O.

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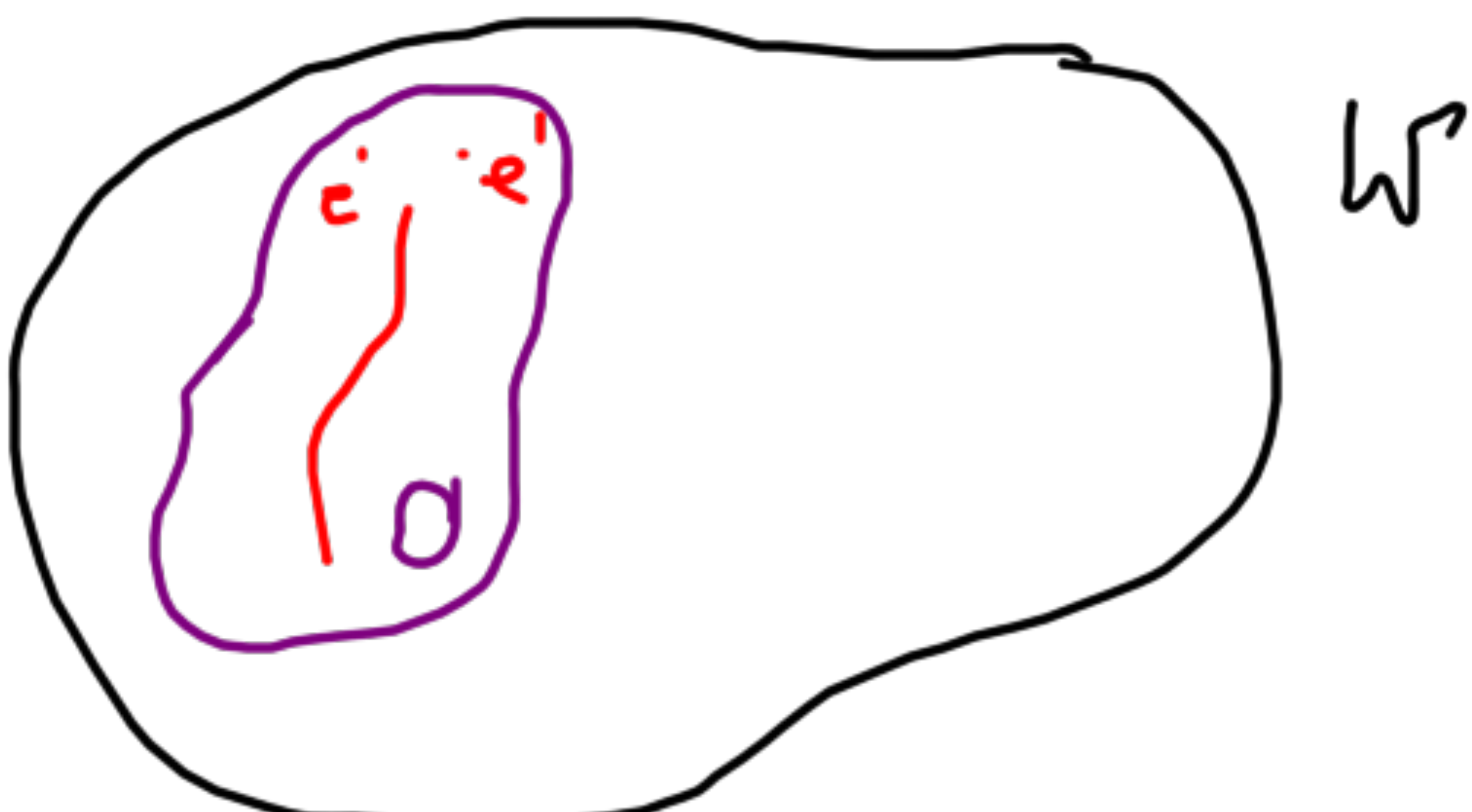

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If there is a splitting pair $\{x, x'\}$, then $\neg\exists z \in W \ (x \preccurlyeq z \wedge x' \preccurlyeq z)$.

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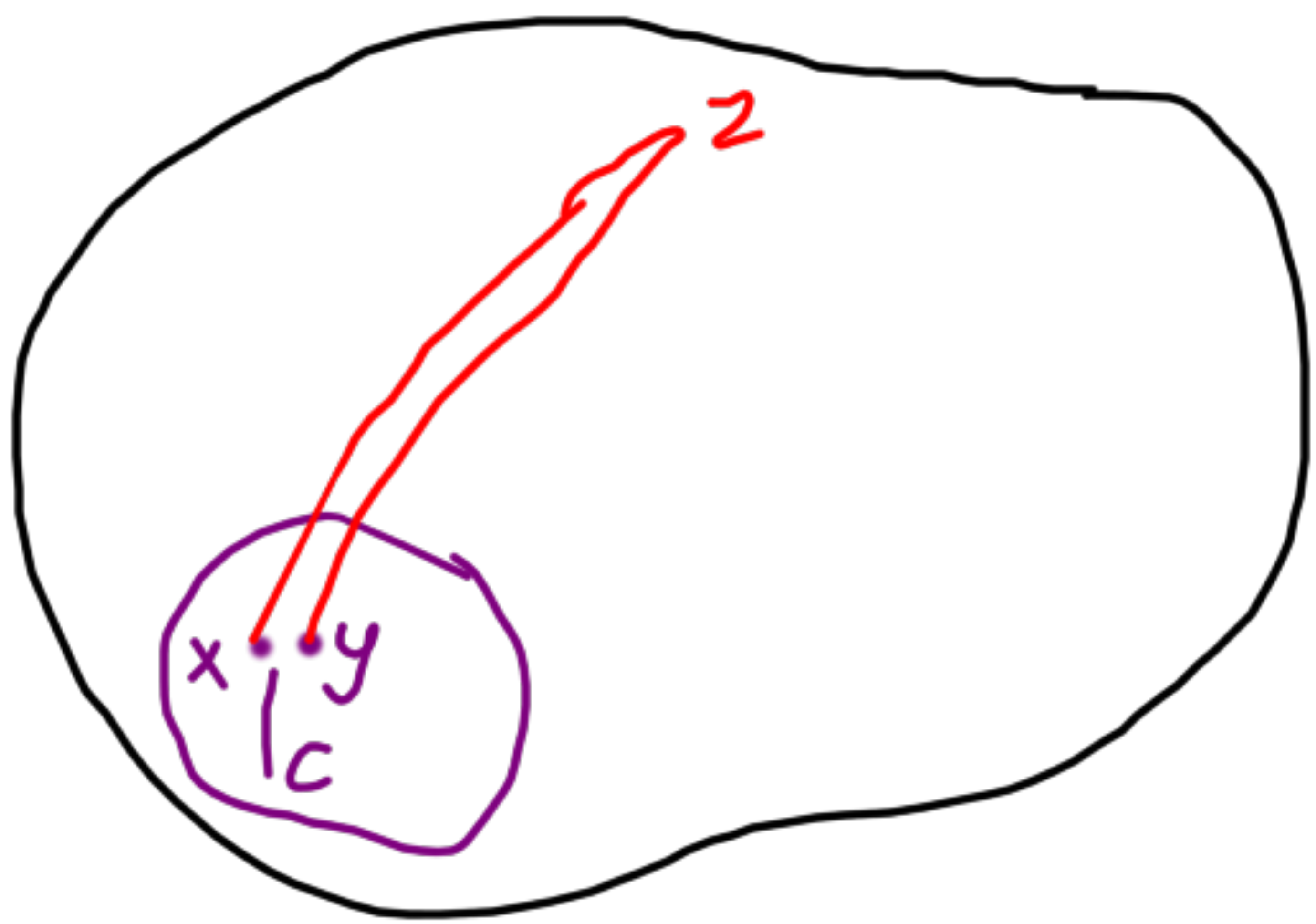

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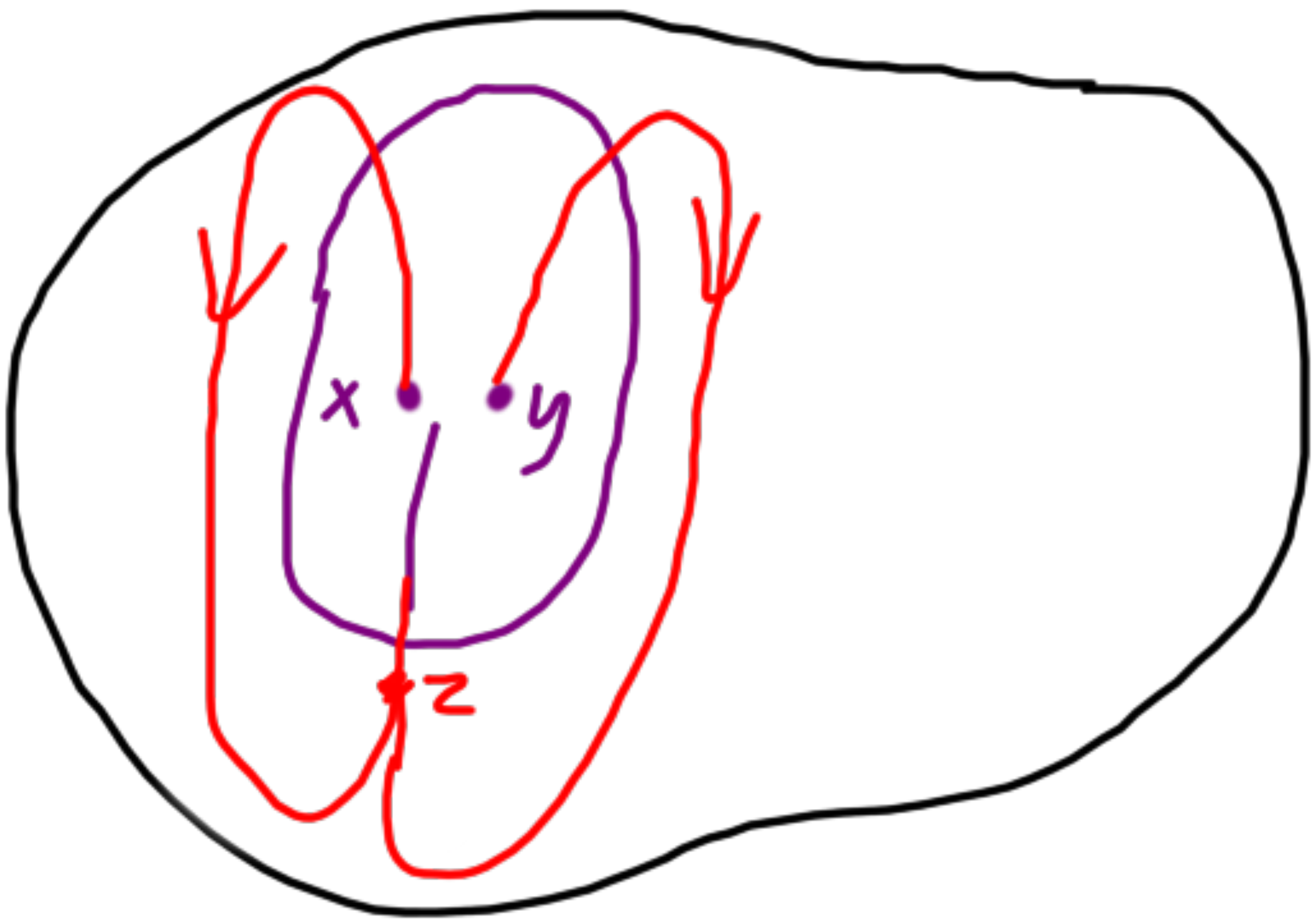

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Gonsistency

$\{e, e'\} \subseteq W$ is consistent iff there is no splitting pair $\{x, x'\}$ such that $x \preceq e$ and $x' \preceq e'$.
$A \subseteq W$ is consistent iff A is pairwise consistent.

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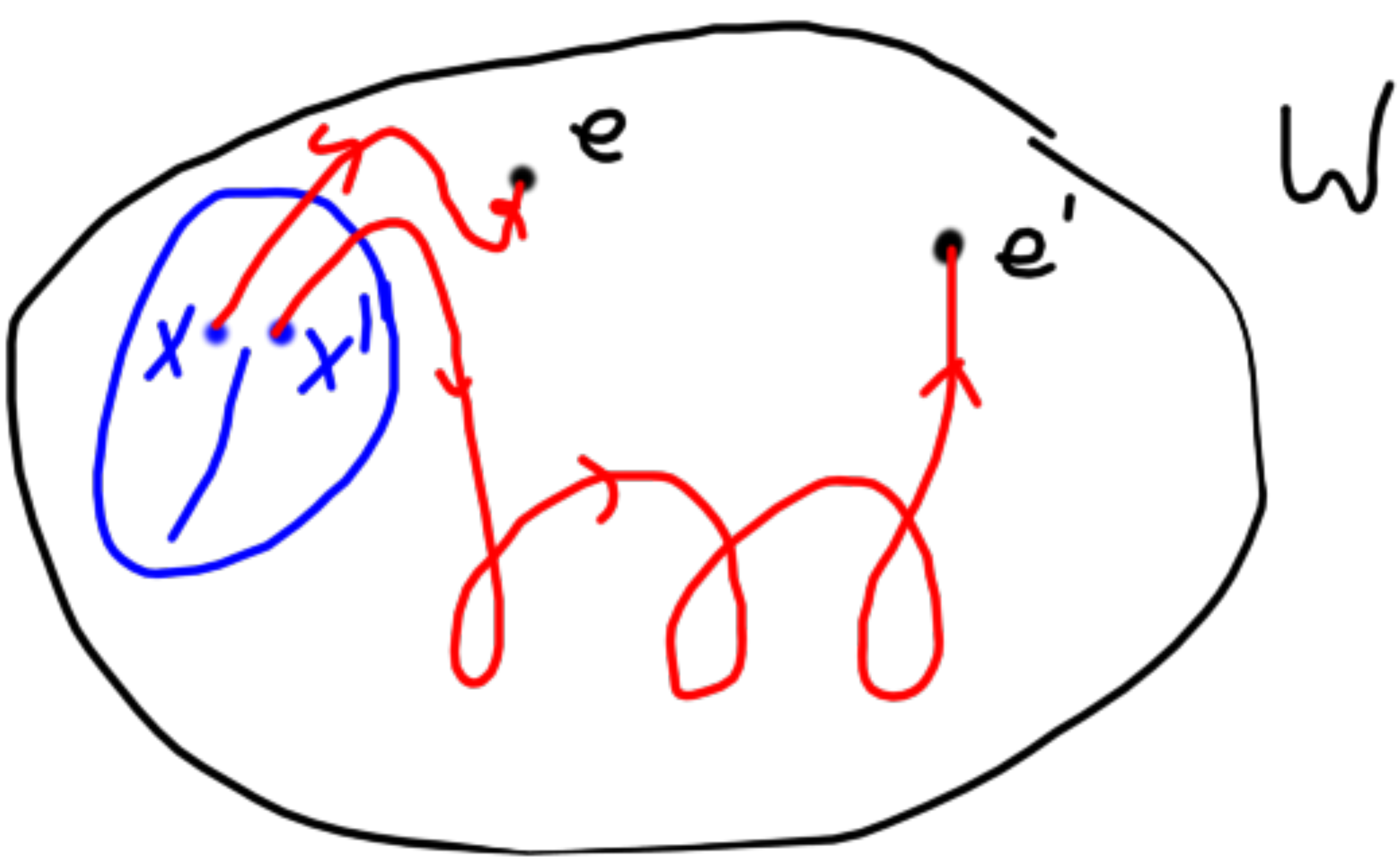

There is at least one maximal pairwise consistent subset of W.

Let A, A' be maximal consistent subsets of W. Then:

(1) A is downward closed.

(2) A has no maximal* and no minimal elements

(3) If $e' \in A' \setminus A$, then there is a "choice pair" $\{x, x'\}$ for A and A', i.e., there is a a chain $C \subseteq A \cap A'$, such that $x = \mathrm{sup}_A(C)$, $x' = sup_{A'}(C)$, $x \neq x'$, and $x' \preceq e'$

(4) If $e, e', e^* \in W$ and $e \preceq e^*$, $e' \preceq e^*$, then there is A s.t. $e, e', e^* \in A$.

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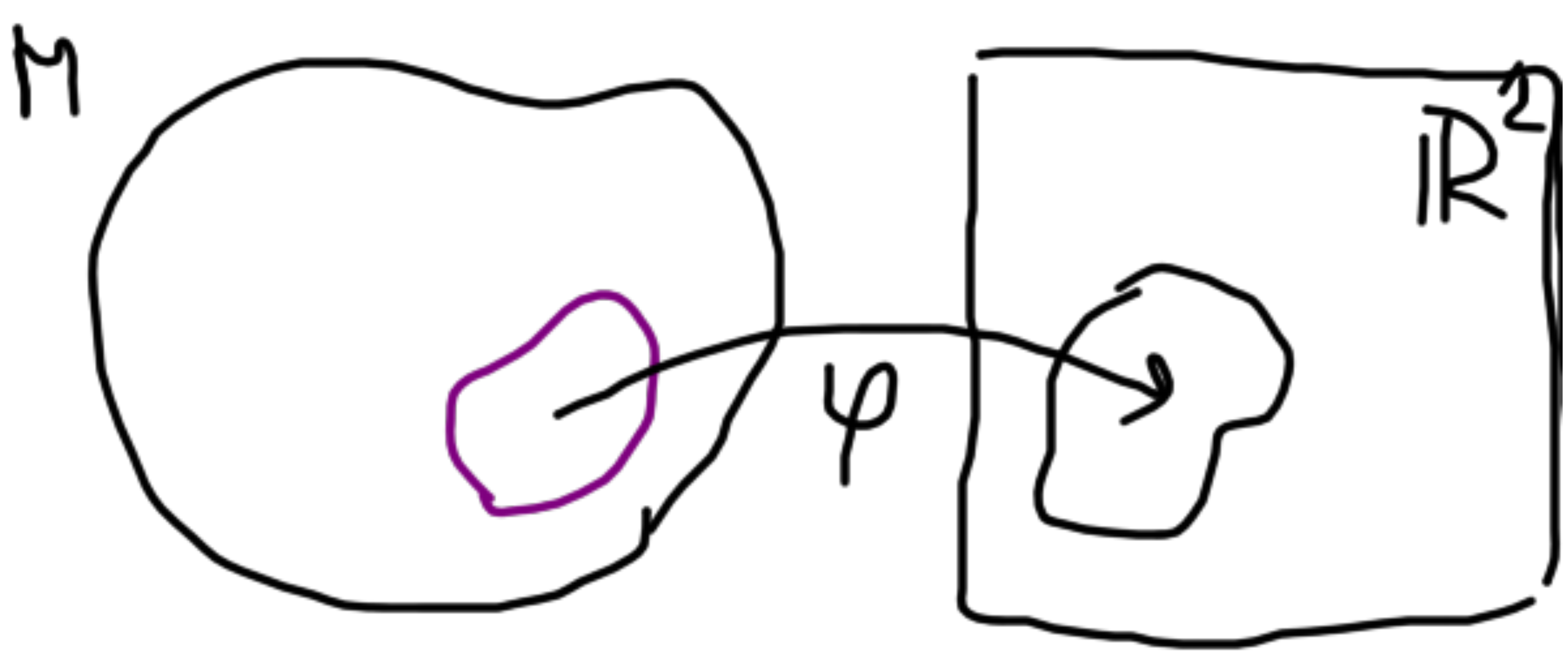
Terminology:
maximal consistent subsets of generalized BST
are called g-histories

g-manifold: putting differential structure on a generalized BST model (generalization of the Geroch-Malament approach)

n-g-chart

An n-g-chart on a model $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$ is a pair $\langle O, \varphi \rangle$, where $O \in \mathcal{O}$ and $\varphi : O \to \mathbb{R}^n$ satisfies, for every $H \in Hist$

If $O \cap H \neq \emptyset$, then
$\varphi_{|H \cap O}$ is injective (i.e., one-to-one),
$\varphi[O \cap H]$ is an open subset of $\mathbb{R}^n$, and
$\forall e, e' \in O \cap H \; e \prec_{|O} e' \leftrightarrow \varphi(e) <_M \varphi(e')$.



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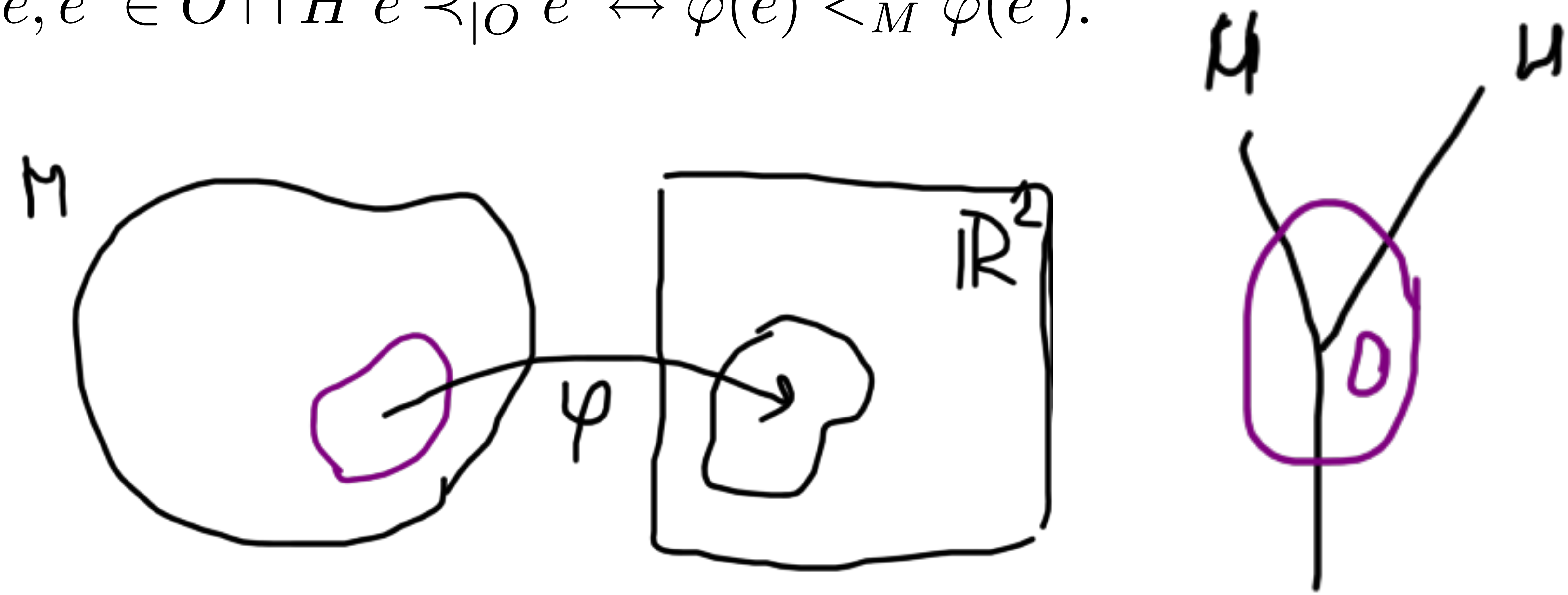

Compatibility of charts

Two n-g-charts $\langle O_1, \varphi_1 \rangle$ and $\langle O_2, \varphi_2 \rangle$ are compatible iff for every $H \in gHist$ either $O_1 \cap O_2 \cap H = \emptyset$ or $O_1 \cap O_2 \cap H \neq \emptyset$ and these two conditions obtain:

(1) $\varphi_i[O_1 \cap O_2 \cap H]$ $(i = 1, 2)$ are open subsets of $\mathbb{R}^n$

(2) $\varphi_2 \varphi_1^{-1} : \varphi_1[O_1 \cap O_2 \cap H] \to \mathbb{R}^n$ and
$\varphi_1 \varphi_2^{-1} : \varphi_2[O_1 \cap O_2 \cap H] \to \mathbb{R}^n$ are smooth.

n-g-manifold

An n-g-manifold is a pair $\langle \mathcal{W}, \mathcal{C} \rangle$, where $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$ is a generalized BST model and $\mathcal{C}$ is a set of n-g-charts on $\mathcal{W}$ satisfying:

(M1) Any two n-g-charts in $\mathcal{C}$ are compatible.

(M2) For every $p \in W$ there is $\langle O, \varphi \rangle \in \mathcal{C}$ such that $p \in O$.

(M3) $\mathcal{C}$ is maximal in the sense that every n-g-chart on $\mathcal{W}$ that is compatible with each n-g-chart in $\mathcal{C}$ belongs to $\mathcal{C}$.

g-manifold topology

Let $\langle \mathcal{W}, \mathcal{C} \rangle$ be a g-manifold on a generalized BST model $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$. We say that $S \subseteq W$ is open in the g-manifold topology, $S \in \mathfrak{T}(W)$, iff

$$\forall p \in S\ \exists\ \langle O, \varphi \rangle \in \mathcal{C}\ (p \in O \wedge O \subseteq S).$$

Local Euclidean?

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If a generalized BST model has a maximal element in the intersection of some two g-histories, then $\mathcal{T}(W)$ is not locally Euclidean.

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If a generalized BST model has a maximal element in the intersection of some two g-histories, then $\mathcal{T}(W)$ is not locally Euclidean.

However, each g-history in a generalized BST model is locally Euclidean in this sense:

for each g-history H, the subspace topology $\mathcal{T}_{\subseteq W}(H)$ is locally Euclidean.

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Maximality

Let H be a g-history in a generalized BST model $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$ and $\langle \mathcal{W}, \mathcal{C} \rangle$ be a g-manifold on $\mathcal{W}$. Then H is a maximal subset of W with respect to being Hausdorff and downward closed.

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Let $\langle \mathcal{W}, \mathcal{C} \rangle$ be a g-manifold on $\mathcal{W} = \langle W, \preceq, \mathcal{O} \rangle$ and $\mathcal{T}(W)$ be its manifold topology.
Then: if A is a maximal subset of W with respect to being Hausdorff and downward closed, then $A \in gHist$.

Importance (if any):
- two constructions of a possible history, via consistency and via maximal Hausdorfness, yield same thing
- spacetime = maximal Hausdorff manifold in a larger thing
- GR-firendly branching.

Special thanks to:

Nuel Belnap
Petr Hoŕava

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END

THANK YOU FOR YOUR ATTENTION

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