Abstract H and S preservation theorems

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Closure Operators

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Let $\mathbb{M}$ be a category, and $Q \subseteq \mathbb{M}$ a class of arrows. For $\mathcal{K} \subseteq \mathrm{Ob}\mathbb{M}$:

$$\overleftarrow{Q}\mathcal{K} := \{A \in \mathrm{Ob}\mathbb{M} : \exists A \overset{Q}{\to} M,\ M \in \mathcal{K}\}$$
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An abstract infinitary logic

Besides $\mathbb{M}$, let a category $\mathbb{S}$ of '***Situations***' be given, connected to $\mathbb{M}$ by $S \to M$ morphisms (called ***interpretations*** of S in M).

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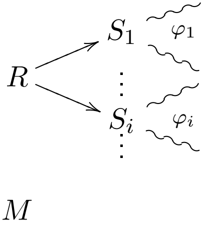

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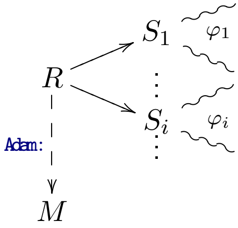

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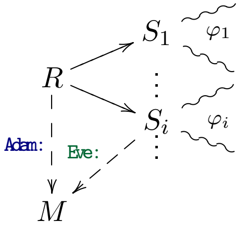

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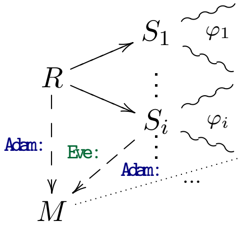

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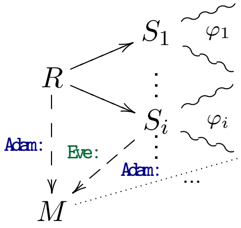

$$M \models \varphi \overset{def}{\Leftrightarrow}$$

Eve can answer every move of Adam
(Eve has *winning strategy*)

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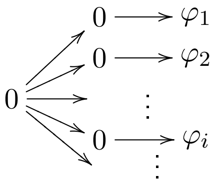

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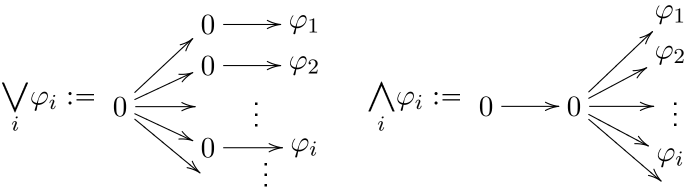

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(Seems particularly useful for *Partial Algebras*. [Burmeister])

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Example (Diagrammatic language of categories)

$$\mathbb{M} := \mathbb{C}\mathrm{at}$$
$$\mathbb{S} := \{\text{graphs with } \textit{commutativity conditions}\}$$
$$\textit{interpretations} := \text{diagrams}$$

[Freyd-Scedrov]

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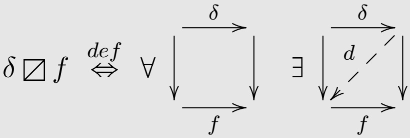

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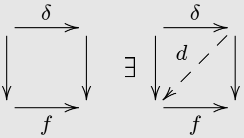

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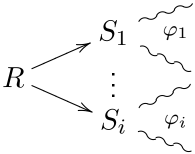

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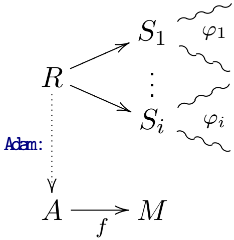

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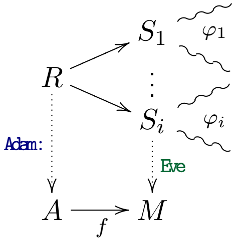

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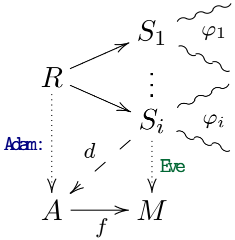

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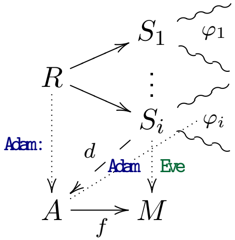

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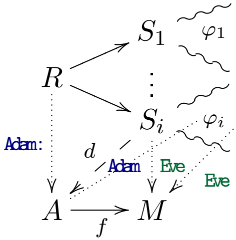

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In that case, for any $\mathcal{K} \subseteq \mathrm{Ob}\mathbb{M}$:

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Preservation theorem

Theorem

Assume the following:

$Q = \boxslash^{\downarrow}(D)$ for some $D \subseteq \mathbb{S}$.

$\mathbb{L}$ has pushouts,

each $M \in \mathrm{Ob}\mathbb{M}$ has a *coreflection* in $\mathbb{S}$ ($S_M \xrightarrow{j} M$) which is also a *reflection* arrow [It implies basically $\mathbb{M} \hookrightarrow \mathbb{S}$]

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for all $\delta \in D$ and $\begin{array}{ccc} \cdot & \xrightarrow{\delta} & \cdot \\ {\scriptstyle\varepsilon}\downarrow & & \\ S_M & & \end{array}$, take their pushout $\begin{array}{ccc} \cdot & \xrightarrow{\delta} & \cdot \\ {\scriptstyle\varepsilon}\downarrow & \ulcorner & \downarrow \\ S_M & \xrightarrow[\sigma_{\delta,\varepsilon}]{} & \cdot \end{array}$

$$\varphi_M := \; 0 \longrightarrow S_M \;\substack{\nearrow \\ \xrightarrow{\sigma_{\delta,\varepsilon}} \\ \searrow}\;\; \substack{\nearrow \\ \xrightarrow{\varrho} \\ \searrow}\; S_M$$

with all right inverses ϱ.

WANTED! *Interpolation theorem*

Looking for conditions for $\Gamma, \Delta \subseteq \mathcal{F}$ to ensure

$$l.u.b.(\mathbf{Mod\,Th}^{\Gamma},\ \mathbf{Mod\,Th}^{\Delta}) = \mathbf{Mod\,Th}^{\Gamma\cap\Delta}$$

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$\mathbf{Mod}\,\mathbf{Th}^{\{\text{Finite fmas}\}} = \mathbf{EeUp}$
with any "nice" $\Delta \subseteq \mathcal{F}$...

Thank you,
Gouranga:)