Residuated Algebras of Binary Relations

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Families of Relations

By a Λ-*family of relations* we mean a collection $\mathfrak{C}$ of binary relations over a base set $U_{\mathfrak{C}}$ such that such that $\mathfrak{C}$ is closed under the operations in Λ. Let $\mathsf{R}(\Lambda)$ denote the class of all Λ-families of relations.

Binary operations: join $+$, meet $\cdot$, relation composition $;$, right $\backslash$ and left $/$ residuals of composition.

Unary operations: converse $\smile$, converse negation $\sim$.

Constants: identity $1'$, zero 0, unit 1.

Operations

The interpretations of the elements of Λ in a Λ-family of relations $\mathfrak{C}$ are as follows. Join $+$ is union, meet $\cdot$ is intersection, zero 0 is the empty set, unit 1 is the universal relation $U_{\mathfrak{C}} \times U_{\mathfrak{C}}$ and

$$
\begin{aligned}
x \,;\, y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\} \\
x \setminus y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (w,u) \in x \text{ implies } (w,v) \in y\} \\
x \,/\, y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (v,w) \in y \text{ implies } (u,w) \in x\} \\
x^{\smile} &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (v,u) \in x\} \\
\sim x &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (v,u) \notin x\} \\
1' &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : u = v\}
\end{aligned}
$$

Special Classes

Assume that ; is definable by Λ and let $\mathfrak{C}$ be a Λ-family of relations. We say that $\mathfrak{C}$ is *commutative* if

$$\mathfrak{C} \models x \,;\, y = y \,;\, x$$

and *dense* if

$$\mathfrak{C} \models x \leq x \,;\, x$$

for all elements x, y of $\mathfrak{C}$, respectively.
The class of commutative and dense Λ-families of relations is denoted by $\mathsf{R}^{cd}(\Lambda)$.
Note that the interpretations of the two residuals coincide in commutative families of relations $\mathfrak{C}$: $\mathfrak{C} \models x \setminus y$ iff $\mathfrak{C} \models y \,/\, x$.

Standard Semantics

We write $\mathfrak{C} \models \tau = \sigma$ iff the interpretation of τ equals the interpretation of σ, for every valuation v into $\mathfrak{C}$.
$\mathfrak{C} \models \tau \leq \sigma$ is defined analogously by interpreting $\leq$ as the subset relation $\subseteq$. Validity will be denoted by $\models$.
An important feature of the (right) residual is the following. We have $x \leq y$ iff $x \setminus y$ contains the identity relation:

$$\mathfrak{C} \models x \leq y \text{ iff } \mathfrak{C} \models 1' \leq x \setminus y \tag{1}$$

for every $\mathfrak{C} \in \mathsf{R}(\Lambda)$.
Note that this makes sense even when $1'$ is not in Λ, since it is meaningful whether $\{(u, u) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}}\}$ is a subset of the interpretation of a Λ-term.

State-Semantics

Let $\mathfrak{C} \in \mathsf{R}(\Lambda)$ for some signature Λ. We define, for every Λ-term τ,

$$\mathfrak{C} \models_s \tau \text{ iff } \mathfrak{C} \models 1' \leq \tau \tag{2}$$

We say that τ is *state-valid* in $\mathsf{R}(\Lambda)$ (in symbols, $\models_s \tau$) iff $\mathfrak{C} \models_s \tau$ for every $\mathfrak{C} \in \mathsf{R}(\Lambda)$.
For instance, $x \setminus y$ is true at (u, u) iff, for every v, (v, u) is in the interpretation of y whenever it is in the interpretation of x.
Cf. PDL.

Connection to Relevance Logic

State-semantics over commutative and dense families of relations is a sound semantics for relevance logic (Dunn, Maddux).

[M, JLC09], [Hirsch-M, RSL11]

The equational theories of $\mathsf{R}^{(cd)}(\Lambda)$ are not finitely axiomatizable for $\{\cdot, +, \backslash\} \subseteq \Lambda \subseteq \{\cdot, +, ;, \backslash, \sim, 1'\}$.

Thus

Relevance logic is not complete w.r.t. (commutative and dense) state-semantics.

Residuated Semigroups

There are (strongly) complete and sound Hilbert-style calculi w.r.t. the state-semantics for

1. $\mathsf{R}(;, \backslash, /)$ and
2. $\mathsf{R}^{cd}(;, \backslash)$.

The proof of [Andréka-M, JoLLI94] Theorem 3.3 goes through with straightforward modifications.

1. For $\mathsf{R}(;, \backslash, /)$ essentially the Lambek Calculus with empty terms, LC_0, works.

$$\models^{(cd)} A_0, \ldots, A_{n-1} \Rightarrow A_n \text{ iff } \models^{(cd)} \Rightarrow (A_0 ; \ldots ; A_{n-1}) \backslash A_n$$
$$\text{iff } \models_s^{(cd)} (A_0 ; \ldots ; A_{n-1}) \backslash A_n$$

2. For $\mathsf{R}^{cd}(;, \backslash)$ we add the following two axioms (corresponding to commutativity and density) to LC_0:

$$A ; B \Rightarrow B ; A \qquad A \Rightarrow A ; A$$

Lower semilattice-ordered residuated semigroups

We define the following Hilbert-style calculus $\vdash_s$. We use the convention that x, y, z may denote empty formulas, while A, B, C must be non-empty formulas. If x is the empty formula, then $x \setminus A$ and $x \,;\, A$ denote A. We have the axioms

$$
\begin{array}{rl}
\text{(Refl)} & A \setminus A \\
\text{(Ass1)} & ((A \,;\, B) \,;\, C) \setminus (A \,;\, (B \,;\, C)) \\
\text{(Ass2)} & (A \,;\, (B \,;\, C)) \setminus ((A \,;\, B) \,;\, C) \\
\text{(ResR)} & ((A \,;\, x) \setminus B) \setminus (x \setminus (A \setminus B)) \\
\text{(Meet1)} & (A \cdot B) \setminus A \\
\text{(Meet2)} & (A \cdot B) \setminus B \\
\text{(Meet3)} & ((A \setminus B) \cdot (A \setminus C)) \setminus (A \setminus (B \cdot C))
\end{array}
$$

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Derivation rules

$$\text{(MP)} \quad \frac{x \setminus A \qquad A \setminus B}{x \setminus B}$$

$$\text{(ResL)} \quad \frac{x \setminus A \qquad (y \,;\, B \,;\, z) \setminus C}{(y \,;\, x \,;\, (A \setminus B) \,;\, z) \setminus C}$$

$$\text{(Mon1)} \quad \frac{x \setminus A \qquad y \setminus B}{(x \,;\, y) \setminus (A \,;\, B)}$$

$$\text{(Mon2)} \quad \frac{x \setminus A \qquad y \setminus B}{(x \cdot y) \setminus (A \cdot B)}$$

$$\text{(Ide1)} \quad \frac{A}{B \setminus (A \,;\, B)}$$

$$\text{(Ide2)} \quad \frac{A}{B \setminus (B \,;\, A)}$$

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If we include the additional axioms

$$\begin{array}{ll} \text{(Comm)} & (A \,;\, B) \setminus (B \,;\, A) \\ \text{(Dens)} & A \setminus (A \,;\, A) \end{array}$$

then the calculus is denoted by $\vdash_s^{cd}$.

The calculus $\vdash_s^{cd}$ is strongly sound and complete w.r.t. state-semantics for $\mathsf{R}^{cd}(\cdot, ;, \setminus)$:

$$\Gamma \vdash_s^{cd} \varphi \text{ iff } \Gamma \models_s^{cd} \varphi$$

for any set Γ of formulas and formula φ.

A similar construction as in the proof of [Andréka-M, JoLLI94] Theorem 3.3 works. We take the Lindenbaum–Tarski algebra $\mathfrak{F}_\Gamma$ of $\vdash_s^{cd}$ and show that $\mathfrak{F}_\Gamma \in \mathsf{R}^{cd}(\cdot, ;, \setminus)$.

Initial Step

By a filter of $\mathfrak{F}_\Gamma$ we mean a subset of elements closed upward (w.r.t. the ordering defined by meet $\cdot$) and under meet. For an element a, let $F(a)$ denote the principal filter generated by a. We will need E, the filter of (the equivalence classes of) Γ-theorems of $\vdash_s^{cd}$, as well.
In the 0th step of the step-by-step construction, we choose distinct u_a, v_a for distinct elements a, and let

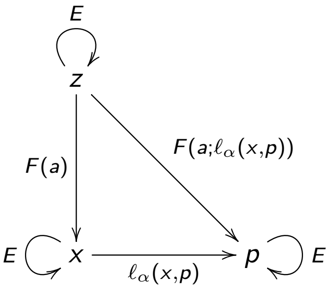
$$\ell_0(u_a, u_a) = \ell_0(v_a, v_a) = E$$
$$\ell_0(u_a, v_a) = F(a)$$

Note that the labels are coherent, e.g., for every $e \in \ell_0(u_a, u_a)$ and $a' \in \ell_0(u_a, v_a)$, we have $e \,;\, a' \in \ell_0(u_a, v_a)$ by (Ide1).

Step for the Residuals

In the $(\alpha+1)$th step we have two subcases. To deal with the residual $\backslash$ we choose a fresh point z, for every point x and element a, and define

$$\ell_{\alpha+1}(z,z) = E$$
$$\ell_{\alpha+1}(z,x) = F(a)$$
$$\ell_{\alpha+1}(z,p) = F(a\,;\ell_\alpha(x,p)) \qquad p \neq x, z$$

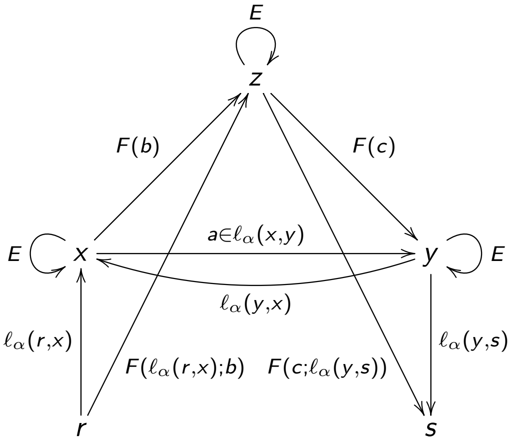


Coherence is easy to check.

Step for Composition

To deal with composition ; we choose a fresh point z, for every $a \in \ell_\alpha(x, y)$ and b, c such that $a \le b\,;c$, and define

$$\begin{aligned}
\ell_{\alpha+1}(z, z) &= E \\
\ell_{\alpha+1}(x, z) &= F(b) \\
\ell_{\alpha+1}(z, y) &= F(c) \\
\ell_{\alpha+1}(r, z) &= F(\ell_\alpha(r, x)\,;b) \qquad r \neq x, z \\
\ell_{\alpha+1}(z, s) &= F(c\,;\ell_\alpha(y, s)) \qquad s \neq y, z
\end{aligned}$$



For instance, we need $c\,;d\,;b \in \ell_{\alpha+1}(z,z)$ for every $d \in \ell_\alpha(y,x)$ (in case $\ell_\alpha(y,x) \neq \emptyset$). By induction, we have that $a\,;d \in \ell_\alpha(x,x)$, i.e., $e \leq a\,;d$ for some $e \in E$. Thus $e \leq (b\,;c)\,;d$ by $a \leq b\,;c$. By commutativity (Comm), we get $e \leq c\,;d\,;b$, whence $c\,;d\,;b \in E = \ell_{\alpha+1}(z,z)$, as desired.

Final Step

Limit step of the construction: take the union of the constructed labelled structures.
After the construction terminates we end up with a labelled structure $(U \times U, \ell)$. We can define a representation of $\mathfrak{F}_\Gamma$ by

$$\mathrm{rep}(a) = \{(u, v) \in U \times U : a \in \ell(u, v)\}$$

Since we used filters as labels, rep respects meet. Injectivity is guaranteed by the 0th step (and the fact that we do not alter labels in later steps). Checking that rep preserves composition and the residual can be done as in the proof of [Andréka-M, JoLLI94] Theorem 3.2.

Upper semilattice-ordered residuated semigroups

Let $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, \smile, 0, 1', 1\}$. Then state-validities for $\mathsf{R}^{cd}(\Lambda)$ and $\mathsf{R}(\Lambda)$ are not finitely axiomatizable.

The heart of the proof is the following.

Andréka-M-Németi, KF12

Let $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, \smile, 0, 1', 1\}$. The equational theory of $\mathsf{R}(\Lambda)$ is not finitely axiomatizable.
Moreover, there is no first-order logic formula valid in $\mathsf{R}(+, ;, \backslash, /, \smile, 0, 1', 1)$ which implies all the equations valid in $\mathsf{R}(+, ;, \backslash, /)$.

Open Problems

Are the (state-)validities for $\mathsf{R}^{cd}(\cdot, ;, \backslash, 1')$ and $\mathsf{R}(\cdot, ;, \backslash, /, 1')$ finitely axiomatizable?