Henkin on Completeness

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August 2012

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3. The set of logical truths is included in the set of theorems of S.
4. The set of logical truths is recursively enumerated (by a calculus S).

Completeness for PL

Some facts from History:

1. Bernays, 1918: "Beiträge zur axiomatischen Behandlung des LogikalKalküs".
2. Post, 1921: "Introduction to a general theory of elementary propositions".
3. Behmann, 1922: "Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem".
4. Bernays, 1926: "Axiomatische Untersuchungen des Aussagen-Kalkuls der "Principia Mathematica" ".
5. Quine, 1938: "Completeness of the Propositional Calculus".

Post's interpretation of the proof

"Our most important theorem gives a uniform method for testing the truth of any proposition of the system; and by means of this theorem it becomes possible to exhibit certain general relations which exists between these propositions. [...] ...this general procedure might be extended to other portions of "Principia", and we hope at some future time to present the begining of such an attempt." [Post 1921, p.164]

"We thus see that given any function the theorem gives a direct method for testing whether that function can or cannot be asserted; and if the test shows that the function can be asserted the above proof will give us an actual method for immediately writting down a formal derivation of its assertion by means of the postulates of Principia." [Post 1921, p.171]

- It is a typical proof based on a critical use of "**normal forms**".
- A K-formula has the following aspect: $(U_n)(E_n)...(U_0)(E_0)A$. That is, a **K-formula** is a formula with a quantificational prefix formed by alternations of universal quantifiers followed by exitential quantifiers.
- The **degree of a K-formula** is defined as the number of such alternations. A K-formula of degree 1 corresponds to $(U_0)(E_0)A$
- The set of $\mathfrak{K}$-**expressions** is formed by the K-formulas of the language.

Normalization proccess

1. **Theorem 3:** If every $\mathfrak{K}$-expression is either refutable or satisfiable, so is every expression.
2. **Theorem 4:** If every expression of degree k is either satisfiable or refutable, so is every expression of degree k+1.
3. **Theorem 5:** Every formula of degree 1 is either satisfiable or refutable.

Introduction of an inductive list of formulas in a propositional language

-$(P)A =_{def} (U_0)(E_0)A$
-$(\forall x_0^1 x_1^1 \ldots x_i^1)(\exists y_0 y_1 \ldots y_j)A$-

$A_1 = A(x_0^1 x_1^1 \ldots x_i^1, y_0 y_1 \ldots y_j)$
$A_2 = A(x_0^2 x_1^2 \ldots x_i^2, y_{j+1} y_{j+2} \ldots y_{2j}) \& A_1$
....
$A_n = A(x_0^n x_1^n \ldots x_i^n, y_{(n-1)j+1} y_{(n-1)j+2} \ldots y_{nj}) \& A_{n-1}$

Some basic facts and the propositional translation

- $(P_n)A_n = (\exists x_0 x_1 ... x_n)A_n$
- **Theorem 6:** $(P)A \rightarrow (P_n)A_n$, for every n.
- B_n: for each A_n we obtain a corresponding propositional formula B_n by replacing the elementary constituens of A_n by different propositional variables.
- A basic fact from PL: each B_n is either satisfiable or refutable (completeness for PL)

Gödel's completeness proof for FOL

Satisfying systems

- **Two cases:** at least one B_n is refutable or no B_n is refutable.
- **Case 1:** At least one B_n is refutable, then $(P_n)A_n$ is refutable too, and by $(P)A \rightarrow (P_n)A_n$, (P)A is refutable
- **Case 2:** No B_n is refutable, i.e. every B_n is satisfiable. Now we obtain satisfiying systems S_i of every level such that each one contains the preceeding systems. From $S_1 ... S_i ...$ we obtain a system S as the sum of $S_1 ... S_i ...$ which satisfies (P)A.
- **Theorem (completeness): Every formula either is refutable or satifiable**
- Compactness is obtained as a corollary from the proof.

Whitehead and Russell, as is well known, constructed logic and mathematics...[...] in a purely formal way (that is, without making further use of the meaning of symbols).

Let us note that the equivalence now proved, "valid=provable", entails, for the decision problem, a reduction of the nondenumerable to the denumerable, since "valid" refers to the totality of functions, while "provable" presupposes only the denumerable totality of formal proofs.

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     "... no matter what (recursive) set of axioms are chosen, the system will contain a formula which is valid but not a formal theorem."
- The *standard semantics* is being determined by structures $\mathfrak{D} = \langle \langle D_\alpha \rangle_{\alpha \in TS}, \ldots \rangle$ where $D_0 = \{T, F\}$, $D_1 \neq \varnothing$, $D_{(0,1)} = \wp(D_1)$,etc.

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Theorem

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- So Henkin defines what he calls *General models* and proves

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- the new method of proof can be generalized

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- An interpretation is build on top of this set using **the set of individuals constants.**

Theorem (Extended Herbrand's)

For any set of sentences $\Gamma \cup \{A\} \subseteq Sent(L)$ *we have:* $\Gamma \vdash A$ *iff* $\Gamma \cup \Delta \vdash_{PL} A$, *where* $\Delta \subseteq Sent(L')$ *effectively given.* $L' = L \cup \mathcal{C}$ *(new individual constants).*

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  2. From propositional interpretation $\Im$ we obtain a first order structure $\mathcal{A}$ such that $\models_{\mathcal{A}} \Gamma$ but $\not\models_{\mathcal{A}} A$ and so, $\Gamma \not\models A$
  3. Thus, $\Gamma \nvdash A$ (soundness FOL)

- We effectively reduce the completeness problem for first order logic to that of sentential logic.

Theorem (Completeness of FOL)

Previous theorem (using completeness of PL) implies completeness of FOL

The completeness of FOL in Henkin's course

Predicate logic: Reduction to sentential logic

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  2. On the other hand, using the structure $\mathcal{A}$ we show that

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- Therefore, $\Gamma \models A$ implies $\Gamma \vdash A$, which is completeness for first order logic.

- Craig had shown the following theorem:

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If A and C are any formulas of predicate logic such that $A \vdash C$, then there is a formula B such that (i) $A \vdash B$ and $B \vdash C$, and (ii) each predicate symbol occurring in B occurs both in A and in C.

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- Henkin's idea was to obtain completeness from a slightly modified version of Craig's theorem.
- *'Notice, however, that if we alter Craig's theorem by replacing the symbol "$\vdash$" with "$\models$" in the hypothesis, but leaving "$\vdash$" unchanged in condition (i) of the conclusion, then the resulting proposition yields the completeness theorem as an immediate corollary.'*

- The main theorem to be proven is:

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Let Γ and Δ any sets of nnf's (negation normal formula) such that $\Gamma \models \Delta$. There is a nnf B such that (i) $\Gamma \vdash B$ and $B \vdash \Delta$, and (ii) any predicate symbol with a positive or negative occurrence in B has an occurrence of the same sign in some formula of Γ and in some formula of Δ.

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- The strong completeness theorem is implied by the previous one. The proof of the theorem is done by contraposition.

Thank you very much