First International Conference on Logic and Relativity (LR12/Németi70)

A Century of Axiomatic Systems for Ordinal Approaches to Special Relativity Theory

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A Century of Axiomatic Systems for Ordinal Approaches to Special Relativity Theory

- What this is not about:

Axioms for Special Relativity Theory:

- I. Space and time are represented by a four-dimensional manifold endowed with a symmetric affine connection $\Gamma^{a}{}_{bc}$ and a metric tensor $\mathbf{g}_{ab}$ satisfying the following:
  - i. $\mathbf{g}_{ab}$ is non-singular with signature -2;
  - ii. $\nabla_c \mathbf{g}_{ab} = 0$;
  - iii. $R^{a}{}_{bcd} = 0$ with $R^{a}{}_{bcd} =_{def} \partial_c \Gamma^{a}{}_{bd} - \partial_d \Gamma^{a}{}_{bc} + \Gamma^{\varepsilon}{}_{bd} \Gamma^{a}{}_{ec} - \Gamma^{e}{}_{bc} \Gamma^{a}{}_{ed}$.
- II. There exist privileged classes of curves exist in the manifold singled out as follows:
  - i. ideal clocks travel along timelike curves and measure the parameter τ (called the proper time)defined by
    $d\tau =_{def} \mathbf{g}_{ab} dx^a dx^b$;
  - ii. free particles travel along timelike geodesics;
  - iii. Light rays travel along null geodesics.

[d'Inverno92 p. 113]

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- What this is <u>not</u> about (continued):

  Axioms for General Relativity Theory:

  - I. Space and time are being represented by a four-dimensional manifold with a symmetric affine connection $\Gamma^a_{\ bc}$ and a metric tensor $\mathbf{g}_{\mathbf{ab}}$ such that:
    - i. $\mathbf{g}_{\mathbf{ab}}$ is non-singular with signature -2;
    - ii. $\nabla_c \mathbf{g}_{\mathbf{ab}} = 0$;
    - iii. $G^{ab} = \kappa T^{ab}$ with $\kappa =_{def} 8\pi G/c^4$.
  - II. Ideal clocks are a priviliged class of curves in the manifold which travel along timelike curves and which measure the parameter τ defined by $d\tau =_{def} \mathbf{g}_{\mathbf{ab}} dx^a dx^b$.

  [d'Inverno92 p. 143, 173]

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- What this is about:
  - Using a single binary primitive relation “after”.
  - This is a partial order relation.
  - Saying that “B is after A” means that “A is in the causal past of B”, or that “A and B are in each others light cones” - there is a “timelike” relation between A and B.
  - Spacelike relations do not need a primitive operator: they can be constructed by combining timelike operators.

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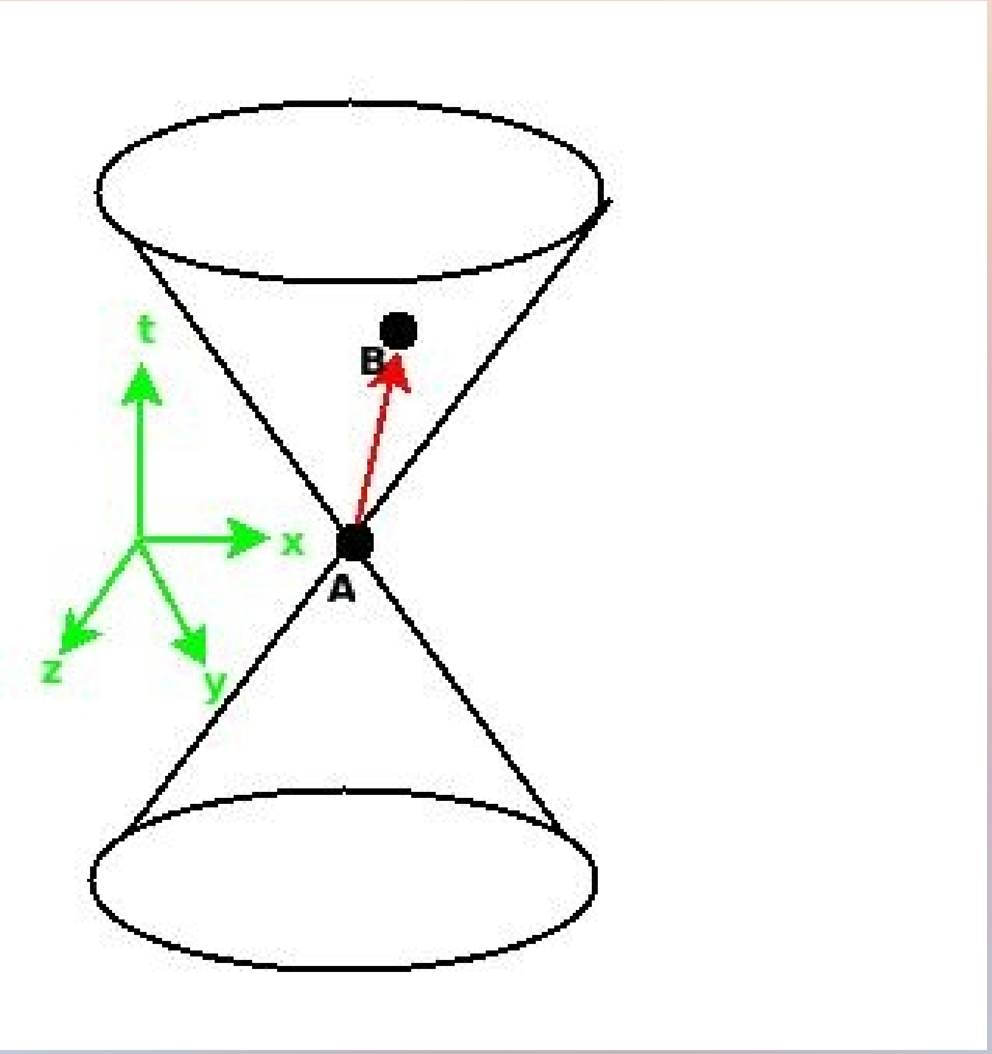

B after A

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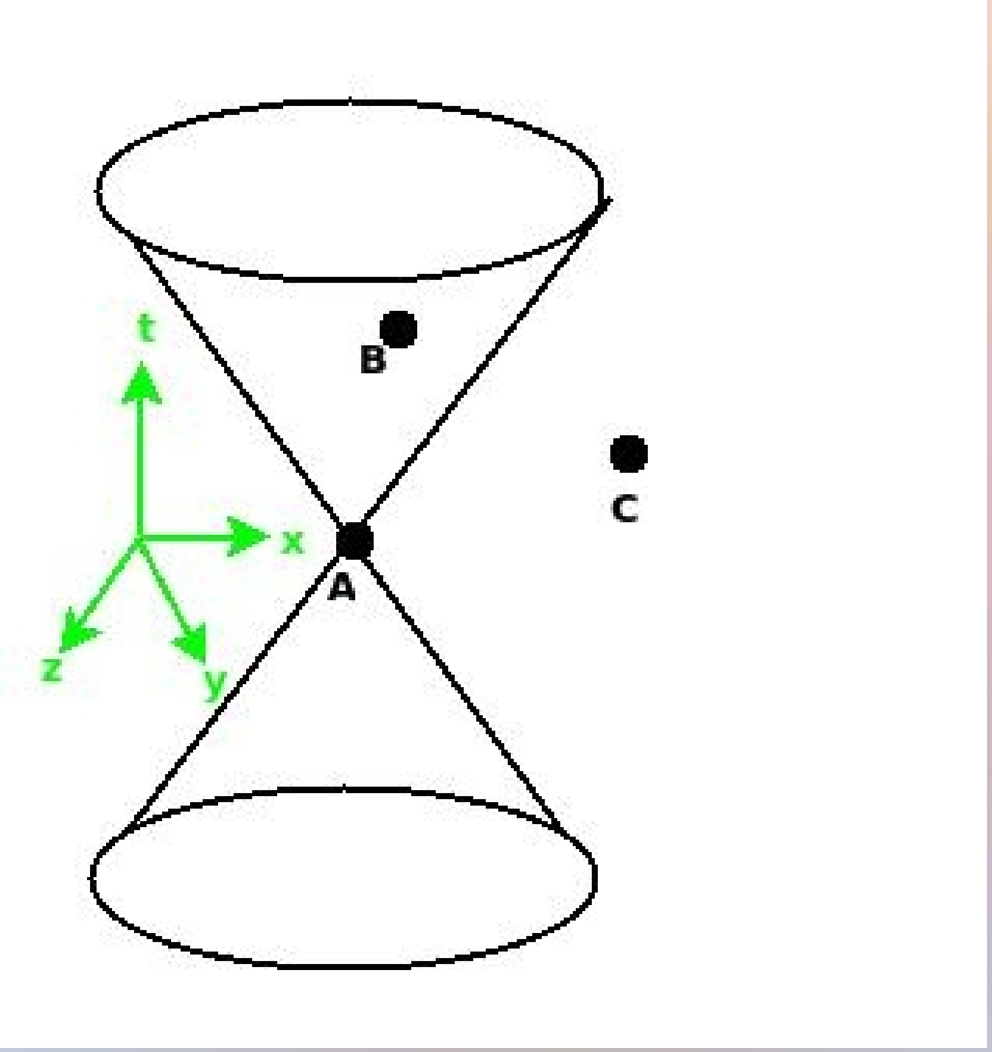

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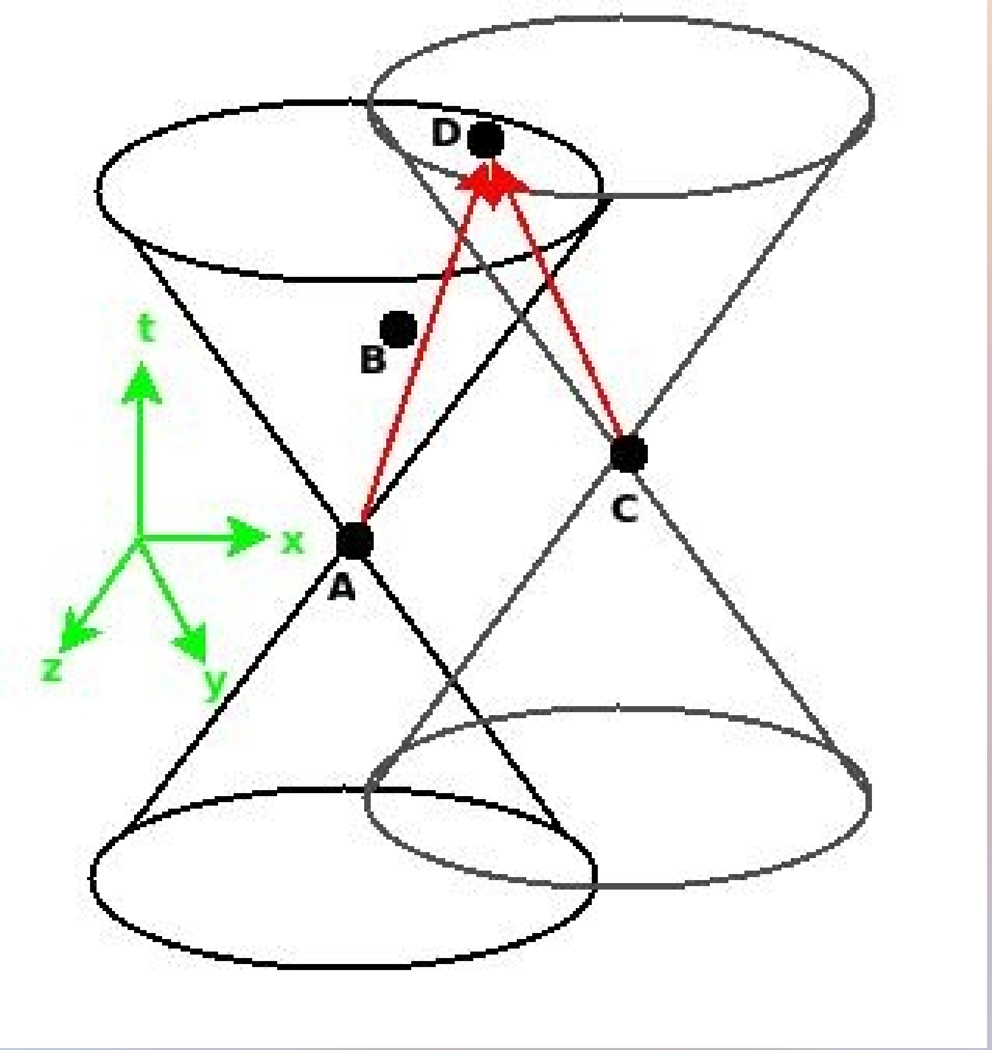

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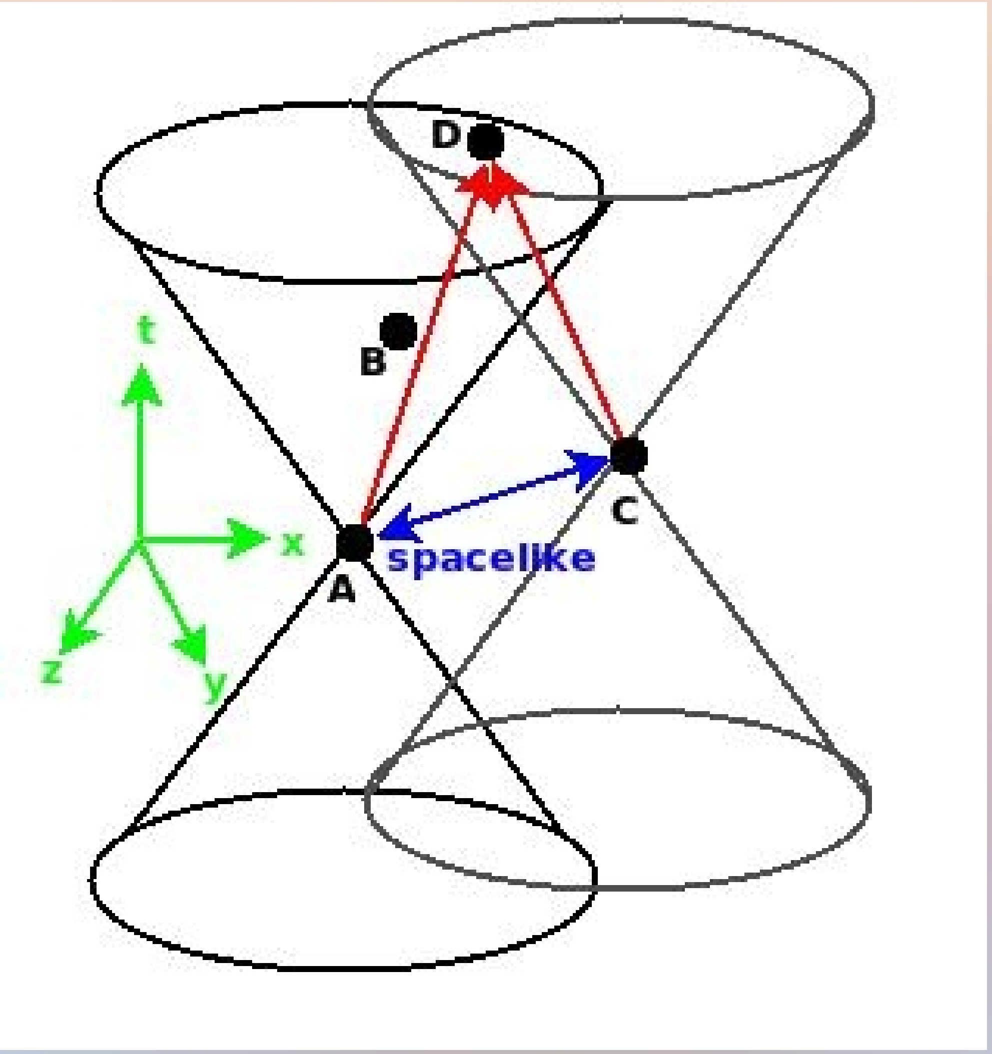

Spacelike relation as a combination of timelike relations

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- Chronology (1/4):
  - 1911: A. A. Robb: “Optical geometry of motion, a new view of the theory of relativity”
  - 1913: A. A. Robb: “A theory of time and space”
  - 1921: A. A. Robb: “The absolute relations of time and space”
  - 1924: H. Reichenbach: “Axiomatik der relativistischen Raum-Zeit-Lehre”
  - 1936: A. A. Robb: “Geometry Of Time And Space”
  - 1948: A. G. Walker: “Foundations of Relativity”

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- Chronology (2/4):
  - 1949: A. D. Aleksandrov: Aleksandrov/Zeeman/Hua theorem
  - 1964: E.C. Zeeman: "Causality implies the Lorentz Group" (Aleksandrov/Zeeman/Hua theorem)
  - 1968: G. Szekeres: "Kinematic geometry; an axiomatic system for Minkowski space-time"
  - 1972: R. W. Latzer: "Nondirected light signals and the structure of time"
  - 1973: J. W. Schutz: "Foundations of Special Relativity - Kinematic Axioms for Minkowski Space-Time"

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- Chronology (3/4):
  - 1974: P. Suppes: “The axiomatic method in the empirical science”
  - 1977: J. A. Winnie: “The Causal Theory of Space-time”
  - 1978: J. Ax: “The elementary foundations of spacetime”
  - 1981: L. K. Hua: Aleksandrov/Zeeman/Hua theorem
  - 1982: A. K. Guts: “Axiomatic relativity theory”
  - 1986: B. Mundy: “Optical Axiomatization of Minkowski Space-Time Geometry”

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- Chronology (4/4):
  - 1987: R. Goldblatt: “Orthogonality and Spacetime Geometry”
  - 1992: N. Belnap: “Branching Space-Time”
  - 1997: J. W. Schutz: “Independent axioms for Minkowski space-time”
  - 2000: N. Belnap: “New foundations for Branching Space-Time”
  - 2002: H. Andréka, J. X. Madarász & I. Németi: “On the logical structure of relativity theories”
  - 2004: H. Andréka, J. X. Madarász & I. Németi: “Logical analysis of relativity theory”

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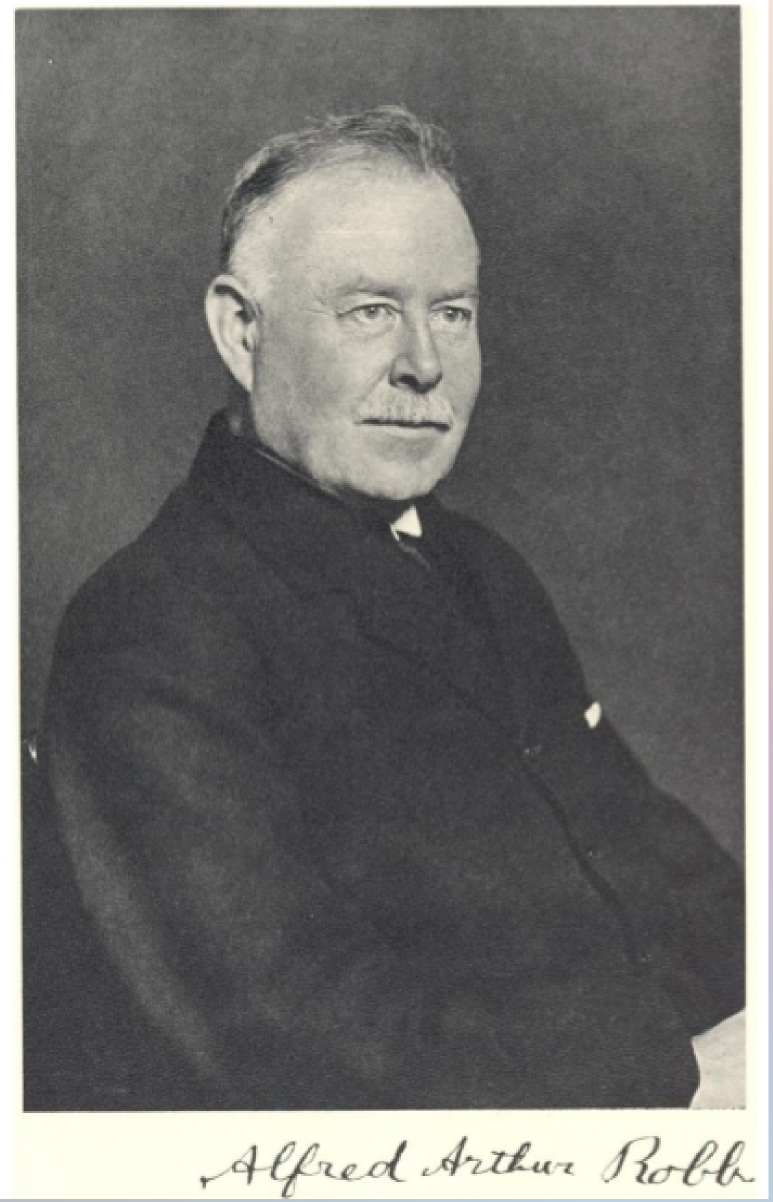
- Alfred Arthur Robb (1873-1936):
  - Study of special relativity in terms of “before” and “after” relations since 1911
  - First axiomatic system in 1913
  - Continued working on this until the end of his life



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- Motivation of A. A. Robb:
  - “From the standpoint of the pure mathematician *Geometry* is a branch of *formal logic*, but there are more aspects of things than one, and the geometrician has but to look at the name of his science to be reminded that it had its origin in a definite *physical* problem.” [Robb11, preface]
  - “The foundations of Geometry have been carefully investigated, especially of late years, by many eminent mathematicians. These investigations have (with the notable exception of those of Helmholtz) been almost all directed towards the Logical aspects of the subject, while the Physical standpoint has received comparatively little attention.

    Speaking of the different "Geometries" which have been devised, Poincaré has gone so far as to say that: "one geometry cannot be more true than another; it can only be more convenient." [...] In reply to this; it must be remembered that the language of Geometry has a certain fairly well defined physical signification which *in its essential features* must be preserved if we are to avoid confusion. [...] It is the contention of the writer that the axioms of Geometry, with a few exceptions, may be regarded as the formal expression of certain Optical facts.” [Robb11, introduction]

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Postulate I. **If an element B be after an element A, then the element A is not after the element B.**

Postulate II. (*a*) **If A be any element, there is at least one element which is after A.**

(*b*) **If A be any element there is at least one element which is before A.**

Postulate III. **If an element B be after an element A, and if an element C be after the element B, the element C is after the element A.**

Postulate IV. **If an element B be after an element A, there is at least one element which is both after A and before B.**

? Postulate V. **If A be any element, there is at least one other element distinct from A which is neither before nor after A.**

Postulate VI. (*a*) **If A and B be two distinct elements, one of which is neither before nor after the other, there is at least one element which is after both A and B, but is not after any other element which is after both A and B.**

(*b*) **If A and B be two distinct elements, one of which is neither after nor before the other, there is at least one element which is before both A and B, but is not before any other element which is before both A and B.**

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Postulate VII. (a) If A_1 and A_2 be elements and if A_2 be a member of α_1, then A_1 is a member of β_2.

(b) If A_1 and A_2 be elements and if A_2 be a member of β_1, then A_1 is a member of α_2.

Postulate VIII. (a) If A_1 be any element and A_2 be any other element in α_1, there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2.

(b) If A_1 be any element and A_2 be any other element in β_1, there is at least one other element distinct from A_2 which is a member both of β_1 and of β_2.

Postulate IX. (a) If a be an optical line and if A_1 be any element which is not in the optical line but before some element of it, there is one single element which is an element both of the optical line a and the sub-set α_1.

(b) If a be an optical line and if A_1 be any element which is not in the optical line but after some element of it, there is one single element which is an element both of the optical line a and the sub-set β_1.

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Postulate X. (a) If α be an optical line and if **A** be any element not in the optical line but before some element of it, there is one single optical line containing **A** and such that each element of it is before an element of α.

(b) If α be an optical line and if **A** be any element not in the optical line but after some element of it, there is one single optical line containing **A** and such that each element of it is after an element of α.

Postulate XI. (a) If $\mathbf{A}_1$ and $\mathbf{A}_2$ be two distinct elements one of which is neither before nor after the other and **X** be an element which is a member both of α_1 and α_2, then there is at least one other element distinct from **X** which is a member both of α_1 and α_2.

(b) If $\mathbf{A}_1$ and $\mathbf{A}_2$ be two distinct elements one of which is neither after nor before the other and **X** be an element which is a member both of β_1 and β_2, then there is at least one other element distinct from **X** which is a member both of β_1 and β_2.

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Postulate XII. (a) If a be an optical line and if **A** be any element not in the optical line but before some element of it, then each optical line through **A**, except the one which intersects a and the one of which each element is before an element of a, has one single element which is neither before nor after any element of a.

(b) If a be an optical line and if **A** be any element not in the optical line but after some element of it, then each optical line through **A**, except the one which intersects a and the one of which each element is after an element of a, has one single element which is neither after nor before any element of a.

Postulate XIII. If two distinct acceleration planes have two elements in common, then any other acceleration plane containing these two elements contains all elements common to the two first-mentioned acceleration planes.

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POSTULATE XIV. (*a*) **If** a **be any inertia line and** $\mathbf{A}_1$ **be any element of the set, then there is one single element common to the inertia line** a **and the sub-set** α_1**.**

(*b*) **If** a **be any inertia line and** $\mathbf{A}_1$ **be any element of the set, then there is one single element common to the inertia line** a **and the sub-set** β_1**.**

POSTULATE XV. **If two general lines, one of which is a separation line and the other is not, lie in the same acceleration plane, then they have an element in common.**

POSTULATE XVI. **If two optical parallelograms lie in the same acceleration plane, then if their diagonal lines of one kind do not intersect, their diagonal lines of the other kind do not intersect.**

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Postulate XVII. **If A_0 and A_x be two elements of an inertia line a such that A_x is after A_0, and if b be a second inertia line which does not intersect a either in A_0, A_x or any element both after A_0 and before A_x, then A_x may be surpassed in a finite number of steps taken from A_0 along a with respect to b.**

Postulate XVIII. **If a, b and c be three parallel inertia lines which do not all lie in one acceleration plane* and A_1 be any element in a and if**

B_1 be the first element in b which is after A_1,

C_1 be the first element in c which is after A_1,

B_2 be the first element in b which is after C_1,

C_2 be the first element in c which is after B_1,

then the first element in a which is after B_2 and the first element in a which is after C_2 are identical.

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POSTULATE XIX. **If P be any optical plane there is at least one element which is neither before nor after any element of P.**

POSTULATE XX. **If W be any optical threefold, then any element of the set must be either before or after some element of W.**

POSTULATE XXI. **If all the elements of an optical line be divided into two sets such that every element of the first set is before every element of the second set, then there is one single element of the optical line which is not before any element of the first set and is not after any element of the second set.**

[Robb14]

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- Robb's axioms: overview
  - 19 postulates hold in 3 dimensions. Postulate XIX adds a dimension, Postulate XX limits the number of dimensions to 4.
  - Not independent: for example, Postulate II can be proven from Postulate V and Postulate VI.
    - “The question as to whether the postulates are all independent is mainly a matter of logical nicety and is of comparatively little importance provided that the number of redundant postulates be not large.” [Robb14 p. 370]

- Robb's axioms: overview (continued)
  - No formal consistency proof:
    - "Since by means of these we have been enabled to set up a coordinate system in the four variables *x, y, z, t*, the question of the consistency of the whole twenty-one postulates is reduced to analysis. It is not proposed to go further into this matter in the present volume, having said sufficient to leave little doubt that they are all consistent with one another." [Robb14, p. 369-370]
  - Second order logic [Suppes74 p. 469]
  - Categorical [Goldblatt87 p.170]

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- Robb's axioms: overview (continued)
  - Representation theorem:
    - “no explicit representation theorem is proved, even though it is evident that this can be done from the results that Robb does prove, and a fortiori, no uniqueness theorem about the representation is proved. The uniqueness theorem was proved in independent and separate fashion much later by Zeeman [Zeeman64] in a beautiful paper that shows Robb’s primitive is adequate for the derivation of the Lorentz transformations.” [Suppes74 p. 468]

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- Robb's axioms: conclusion
  - “[Robb's] approach has never gained much popularity, perhaps because many of the twenty-one postulates he used seem ad hoc, introduced only as needed to allow the technical development of the theory to proceed. These postulates generally lack physical significance, or even a more general geometrical intuitiveness, and as Suppes [1973] observes, “the complexity of the axioms stand in marked contrast to the simplicity of his single primitive concept.””

    [Goldblatt87 p. 170]

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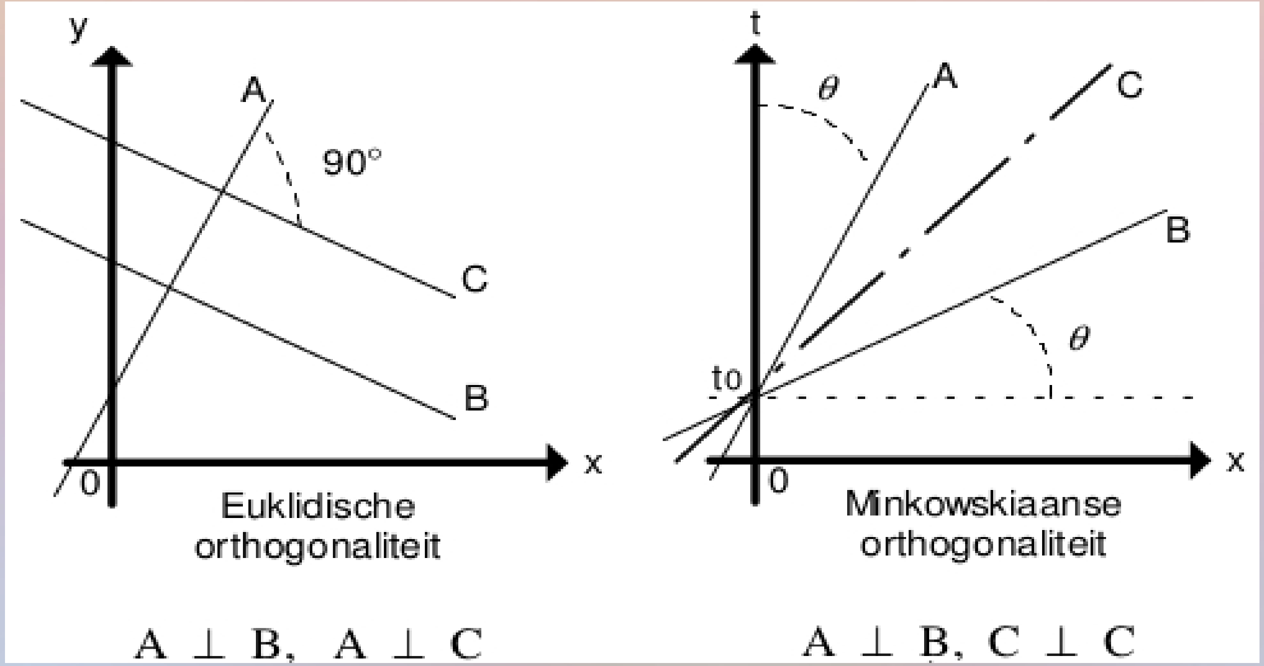
- Robert Goldblatt 1987:

  “Orthogonality and Spacetime Geometry”



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- Goldblatt uses a ternary "betweenness" relation as a primitive to construct an affine space, after that he uses orthogonality to make this space metric. By means of an axiom which excludes singular lines and two axioms on null lines, this metric space becomes Minkowskian.
- He shows that the betweenness relation and orthogonality can be constructed from a single "after" relation, using an auxiliary relation $\ll$ which is the “after” relation on light rays:

$$x \ll y \Leftrightarrow \exists z[z \neq x \wedge \sim(z \text{ after } x \vee x \text{ after } z) \wedge y \text{ after } x \wedge y \text{ after } z \wedge \forall u(u \text{ after } x \wedge u \text{ after } z \Rightarrow \sim y \text{ after } u)]$$

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- $x\lambda y$ is an abbreviation for $x \neq y$ & $xy \perp xy$ which expresses that there exists a lightlike relation between x and y. Goldblatt shows that: $x\lambda y \Leftrightarrow (x \ll y) \vee (y \ll x)$
- L(tzu) is an abbreviation for "t is collinear with x and y" and can be defined in terms of betweenness: $B(xyz) \vee B(yzx) \vee B(zxy)$.
- $P(x_1x_2x_3z)$ is an abbreviation for "z is coplanar with x_1, x_2 and x_3" and is defined in a similar way.
- Using L(tzu) and $P(x_1x_2x_3z)$, the abbreviations $TF(x_1\ldots x_5)$ for threefold and $FF(x_1\ldots x_5z)$ for fourfold are defined.
- $xy \parallel zw$ is an abbreviation for

  $$P(xyzw) \wedge [\forall u(L(xyu) \Rightarrow L(zwu)) \vee \forall u(L(xyu) \Rightarrow \sim L(zwu))]$$

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- Goldblatt's axioms for affine space:
  - **1.** $B(xyx) \Rightarrow x=y$
  - **2.** $B(xyz) \wedge B(yzu)\ y \neq z \Rightarrow B(xyu)$
  - **3.** $B(xyz) \wedge B(xyu) \wedge x \neq z \Rightarrow B(yzu) \vee B(yuz)$
  - **4.** $\exists x$: $[B(xyz)\ \&\ x \neq y]$
  - **5.** $B(xtu) \wedge B(yuz) \Rightarrow\ \exists v{:}[B(xvy) \wedge B(ztv)]$
  - **6.** $\exists x_1 \ldots \exists x_5{:}[\sim TF(x_1 \ldots x_5) \wedge \forall z{:}FF(x_1 \ldots x_5 z)]$
  - **7.** $B(xut) \wedge B(yuz) \wedge x \neq u \Rightarrow$
    $\exists v \exists w{:}[B(xyv) \wedge B(xzw) \wedge B(vtw)]$
  - **8.** $\exists z \forall x \forall y{:}[\varphi \wedge \psi \Rightarrow B(zxy)] \Rightarrow \exists u \forall x \forall y{:}[\varphi \wedge \psi \Rightarrow B(xuy)]$
    where φ is any formula with x free, but not y, z or u, while ψ has y free but not x, z or u.

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- The following axioms are added to those of the affine space to make it metric:
  - **OS1.** $xy \perp zw \Rightarrow zw \perp xy$
  - **OS2.** $[P(xyzw) \Rightarrow xy \perp zw] \vee$

    $\exists t[P(xyzt) \wedge \forall u(P(xyzu) \Rightarrow (xy \perp zu \Leftrightarrow L(tzu)))]$
  - **OS3.** $[xy \perp zw \wedge xz \perp yw] \Rightarrow xw \perp yz$
  - **OS4.** $[xy \perp yw \wedge xy \perp yz \wedge P(ywzu)] \Rightarrow xy \perp yu$
  - **OS5.** $[xy \perp zw \wedge zw \| uv] \Rightarrow xy \perp uv$

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- To the metric affine space, the following axioms are added to make it Minkowskian:
  - **M1.** $\forall x \forall y \exists w \sim(xy \perp yw)$
  - **M2.** $\exists x \exists y (x \lambda y)$
  - **M3.** $[x\lambda y \wedge z\lambda w \wedge xy \perp zw] \Rightarrow xy \| zw$

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- Discussion of Goldblatt's system:
  - First order logic, but there are an infinite number of continuity axioms which are generated by an axiom schema *à la Tarski*.
  - Complete & decidable.
  - Not categorical.
  - Not independent.
  - Representation theorem: a version of the Aleksandrov/Zeeman/Hua theorem is proven to get to the Lorentz transformations.

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- John W. Schutz 1997:
  "Independent axioms for Minkowski space-time"
- Undefined primitive basis:
  - Ternary betweenness relation [abc]
  - Set of events: ε
  - Set $\mathcal{P}$ of subsets of ε, elements of $\mathcal{P}$ are called "paths"
  - Minkowski space-time $\mathcal{M} = \langle ε, \mathcal{P}, [\ldots] \rangle$
- Logic, set theory and arithmetic of real numbers are presupposed.

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- Axioms of Order:
  - **O1.** For events a,b,c$\in \varepsilon$: [abc] $\Rightarrow \exists Q \in \mathcal{P}$:a,b,c $\in$ Q
  - **O2.** For events a,b,c$\in \varepsilon$: [abc] $\Rightarrow$ [cba]
  - **O3:** For events a,b,c$\in \varepsilon$: [abc] $\Rightarrow$ a,b,c are distinct
  - **O4.** For distinct events a,b,c,d$\in \varepsilon$: [abc] $\wedge$ [bcd] $\Rightarrow$ [abd]
  - **O5.** For any path Q$\in \mathcal{P}$ and any 3 distinct events a,b,c$\in$ Q:

    [abc] $\vee$ [bca] $\vee$ [cab] $\vee$ [cba] $\vee$ [acb] $\vee$ [bac]

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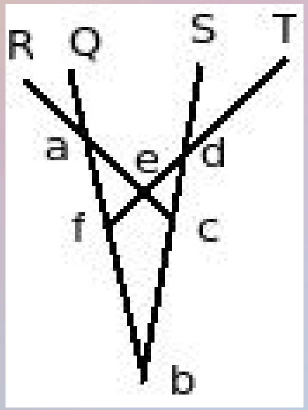
- Axioms of Order (continued):
  - Definition: A sequence of events Q_0, Q_1, Q_2, … (of a path Q) is called a *chain* if:
    - it has two distinct events, or
    - it has more than two distinct events for all $i \geq 2$, $[Q_{i-2}\ Q_{i-1}\ Q_i]$
  - **O6.** If Q,R,S are distinct paths which meet at events $a \in Q \cap R$, $b \in Q \cap S$, $c \in R \cap S$ and if:
    - There is an event $d \in S$ shuch that [bcd], and
    - There is an event $e \in R$ and a path T which passes through both d and e such that [cea],



then T meets Q in an event f which belongs to a finite chain [a..f..b].

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- Some definitions needed for the axioms of incidence:
  - A SPRAY is a set of paths which meet at a given event:
    - SPR[x] := {R: x∈R, R}
  - A spray is a set of events which meet at a given event:
    - spr[x] := {R_y: R_y∈R, R ∈ SPR[x]}
  - A SPRAY is a 3-SPRAY if:
    - It contains four independent paths, and
    - All paths of the SPRAY are dependent of these four paths.
  - Q(b,∅) := {x: there is no path which contains b and x, x∈Q}

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- Axioms of incidence:
  - **I1.** ε is not empty.
  - **I2.** For any two distinct events $a,b \in \varepsilon$ there are paths R,S such that $a \in R$, $b \in S$ and $R \cap S \neq \varnothing$.
  - **I3.** For any two distinct events, there is at most one path which contains both of them.
  - **I4.** If $\varepsilon \neq \varnothing$, then there is at least one 3-SPRAY.
  - **I5.** For any path Q and any event $b \notin Q$, the unreachable set $Q(b,\varnothing)$ contains (at least) two events.

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- Axioms of incidence (continued)
  - **I6.** For any path Q, any event $b \notin Q$, and distinct events $Q_x, Q_z \in Q(b,\varnothing)$, there is a finite chain $[Q_0 \ldots Q_n]$ with $Q_0 = Q_x$ and $Q_n = Q_z$, such that for all $i \in \{1,2,\ldots,n\}$
    - $Q_i \in Q(b,\varnothing)$
    - $[Q_{i-1}\ Q_y\ Q_i] \Rightarrow Q_y \in Q(b,\varnothing)$
  - **I7.** Given any path Q, any event $b \notin Q$ and events $Q_x \in Q \backslash Q(b,\varnothing)$ and $Q_y \in Q \backslash Q(b,\varnothing)$, there is a finite chain

    $[Q_0 \ldots Q_m \ldots Q_n]$ with $Q_0 = Q_x$, $Q_m = Q_y$ and $Q_n \in Q \backslash Q(b,\varnothing)$.

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- Definition of unreachable subset of Q from Q_a via R:
  - $Q(Q_a,R,x,\varnothing) := \{Qy: [x\ Q_y\ Q_a]$ and $\exists R_w \in R$ such that

    $$Q_a,\ Q_y \in Q(R_w,\varnothing)\}$$

- Axiom of Symmetry or Isotropy:
  - **S.** If Q, R, S are distinct paths which meet at some event x and if $Q_a \in Q$ is an event distinct from x such that $Q(Q_a,R,x,\varnothing)=Q(Q_a,S,x,\varnothing)$ then
    - there is a mapping $\theta: \varepsilon \to \varepsilon$
    - which induces a bijection $\Theta: \mathcal{P} \to \mathcal{P}$

    such that

    - the events of Q are invariant, and
    - $\Theta: R \to S$.

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- Definitions of bounds:
  - Given a path $Q \in \mathcal{P}$ and an infinite chain $[Q_0\ Q_1\ \ldots]$ of events in Q, the set $\mathcal{B} = \{Q_b : i<j\ [Q_i\ Q_j\ Q_b];\ Q_i, Q_j, Q_b \in Q\}$ is called the set of bounds of the chain.
  - If there is a bound Q_b such that for all $Q_{b'} \in \mathcal{B} \setminus \{Q_b\}$, $[Q_0\ Q_b\ Q_{b'}]$ we say that Q_b is a closest bound.
- Axiom of Continuity
  - **C.** Any bounded infinite chain has a closest bound.

- Discussion of Schutz' axioms:
  - Second order because of Continuity axiom
  - Categorical
  - Consistent if the theory of Real Numbers is consistent
  - Independence is proven using independence models
  - Representation theorems

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- Nuel Belnap: Branching Space-Time (BST):
  - Branching Time: at certain times, things can go different ways and their outcomes may be incompatible.
  - Belnap intended to show that space-time is not incompatible with indeterminism, going against the misconception that in a four-dimensional “block space-time” is necessarily deterministic.

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- Primitive notions of BST:
  - Our World is a set of point events.
  - Postulate 1: The relation $\leq$ is a nontrivial partial ordering of Our World:
    - Nontriviality: Our World is not empty
    - Reflexivity: $e \leq e$
    - Transitivity: if $e_1 \leq e_2$ and $e_2 \leq e_3$ then $e_1 \leq e_3$
    - Antisymmetry: if $e_1 \leq e_2$ and $e_2 \leq e_1$ then $e_1 = e_2$

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- Some definitions:
  - A chain is a subset of Our World all members of which are comparable by ≤: for e_1, e_2 in the chain, either $e_1 \leq e_2$ or $e_2 \leq e_1$.
  - A subset E of Our World is directed just in case for all e_1 and e_2 in E there is a point event e_3 in E that is their common upper bound: $e_3 \in E$ and $e_1 \leq e_3$ and $e_2 \leq e_3$.
  - A subset h of Our world is a history just in case h is a maximal directed subset of Our World: h itself is a directed subset of our World, and no proper superset of h has this feature.

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- Postulate 13: Historical connection
  - Every pair of histories has a nonempty intersection.
- Definition:
  - A Minkowski branching space-time is a model of Our World in which each history is a Minkowski space-time (in the standard sense found in the litterature).
- Postulate 26: Choice principle
  - For each two histories, there is at least one choice point.

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- Postulate 28: Prior Choice Principle, point event version
  - If e belongs to h_1-h_2, then there is a choice point for h_1 and h_2 lying in the past of e.
- There are some more postulates in an appendix, extending the choice principle from events to chains of events.
- A more symbolic rigorous axiomatization was introduced in [Belnap2000], but this is unpublished and has since been remove from Belnap's website.

- Discussion of BST:
    - This is a *protophysical* theory. BST has no notion of metric, but a metric can be introduced separately, for example in a “Minkowski BST”.

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- SpecRel:
  - Has been discussed extensively by other speakers
  - The only thing which I can imagine which would be more absurd than me explaining this, is if I would try to teach you the Hungarian language.
  - Some remarks:
    - First order logic, complete & consistent, decidable, representation theorem.
    - Lego block analogy: SR, SpecRel, SpecRel*, AccRel, GenRel, etc...
    - A categorical variant exists, as discussed in [Andréka04], the price is losing decidability.
  - Many happy & fruitful years to prof. István Németi.

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- Conclusions:
  - Special Relativity Theory and Minkowski space-time can be defined using only a partial order relation.
  - Depending on what the author is interested in, we get different kinds of axiomatic systems:
    - Except for BST, all systems have a metric.
    - Most systems are of second order, the exceptions are Goldblatt's system and SpecRel from the Hungarian School.
    - We have to chose between being decidable and being categorical.

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