Approximating the 2-variable fragment of predicate logic with propositional modal logics: a survey of recent results

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Propositional bimodal logics

- **bimodal formulas:** $\varphi \;=\; p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_0\varphi \mid \Diamond_1\varphi \mid \Box_0\varphi \mid \Box_1\varphi$
- **relational structures:** $\mathfrak{F} = \langle W, R_0, R_1 \rangle$
- **models:** $\mathfrak{M} = \langle \mathfrak{F}, \nu \rangle$, where $\nu : \textit{Variables} \to 2^W$
  - $(\mathfrak{M}, w) \models p$ iff $w \in \nu(p)$
  - $(\mathfrak{M}, w) \models \neg\varphi$ iff $(\mathfrak{M}, w) \not\models \varphi$
  - $(\mathfrak{M}, w) \models \varphi \vee \psi$ iff $(\mathfrak{M}, w) \models \varphi$ or $(\mathfrak{M}, w) \models \psi$
  - $(\mathfrak{M}, w) \models \Diamond_i\varphi$ iff $\exists v \,\big(wR_iv$ and $(\mathfrak{M}, v) \models \varphi\big)$
  - $(\mathfrak{M}, w) \models \Box_i\varphi$ iff $\forall v \,\big($if wR_iv then $(\mathfrak{M}, v) \models \varphi\big)$
- **validity:** $\mathfrak{F} \models \varphi$ iff $(\mathfrak{M}, w) \models \varphi$, for all $\mathfrak{M} = \langle \mathfrak{F}, \nu \rangle$ and $w \in W$

Modal logics vs. Boolean Algebras with Operators

- modal operator $\Diamond_i$
- formula φ
- relational structure $\mathfrak{F}$
- $\mathfrak{F} \models \varphi$
- **Logic_of** $\mathcal{C} = \{\varphi : \mathfrak{F} \models \varphi, \forall \mathfrak{F} \in \mathcal{C}\}$
  - finitely axiomatisable
  - decidable
  - finite model property

- unary function symbol $f_{\Diamond_i}$
- term t_φ
- full complex algebra $Cm(\mathfrak{F})$ of $\mathfrak{F}$
- $Cm(\mathfrak{F}) \models (t_\varphi = 1)$
- **EqTheory_of Var**$\{Cm(\mathfrak{F}) : \mathfrak{F} \in \mathcal{C}\}$
  - finitely based
  - decidable
  - finite algebra property

2D cylindric set algebras

Subalgebras of the full complex algebra of 2D **'square-structures'** of the form

$$\langle\, W \times W, \equiv_0, \equiv_1, D_{01} \,\rangle$$

where, for all $\langle x, y\rangle$, $\langle x', y'\rangle \in W \times W$,

$$\begin{array}{lll} \langle x, y\rangle \equiv_0 \langle x', y'\rangle & \text{iff} & y = y' \\ \langle x, y\rangle \equiv_1 \langle x', y'\rangle & \text{iff} & x = x' \\ D_{01} = \{\langle x, x\rangle \mid x \in W\} & & \end{array}$$

$$\mathbf{RCA_2} = \mathsf{Var}\{Cm\,\langle\, W \times W, \equiv_0, \equiv_1, D_{01} \,\rangle \mid W \text{ is a nonempty set}\}$$

Modal logic (or RCA_2) vs. two-variable predicate logic

multimodal formula φ $\rightsquigarrow$ f-o formula φ^* with ≤ 2 variables

$$\begin{array}{rcl} p & \mapsto & P(x,y) \\ \neg\varphi & \mapsto & \neg\varphi^* \\ \varphi \vee \psi & \mapsto & \varphi^* \vee \psi^* \\ \Diamond_0\varphi & \mapsto & \exists x\ \varphi^* \\ \Box_0\varphi & \mapsto & \forall x\ \varphi^* \\ \Diamond_1\varphi & \mapsto & \exists y\ \varphi^* \\ \Box_1\varphi & \mapsto & \forall y\ \varphi^* \\ \delta & \mapsto & x = y \end{array}$$

φ is valid in 'square-structures' $\Longleftrightarrow$ φ^* is f-o valid

2D propositional approximations of $\mathbf{RCA_2}$

We classify the properties of $\mathbf{RCA_2}$:

- **Group (0):** BAO properties
- **Group (1):** additional properties of the **individual** cylindrifications

$x \le c_i x$	$p \to \Diamond_i p$	reflexive
$c_i c_i x \le c_i x$	$\Diamond_i \Diamond_i p \to \Diamond_i p$	transitive
$x \le -c_i - c_i x$	$p \to \Box_i \Diamond_i p$	symmetric

- **Group (2):** **'dimension-connecting'** properties due to the 2D structure

$c_0 c_1 x = c_1 c_0 x$	$\Diamond_0 \Diamond_1 p \leftrightarrow \Diamond_1 \Diamond_0 p$	commutativity
	Henkin axiom	

What's next? We try to keep (0) and (2), but relax (1)

$\rightsquigarrow$ 2D structures with possibly **'sub-equivalence'** relations

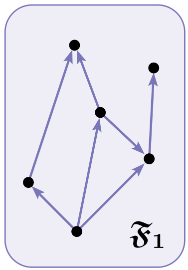
Diagonal-free case: 2D relational structures

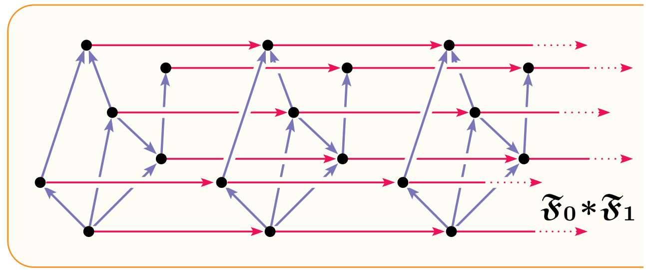
Segerberg 1973, Shehtman 1978, Gabbay-Shehtman 1998

For $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ and $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$,
the **$*$-product structure** is $\mathfrak{F}_0 * \mathfrak{F}_1 = \langle W_0 \times W_1, R_h, R_v \rangle$, where

$\langle x, y \rangle \, R_h \, \langle x', y' \rangle$ iff
$y = y'$ and $x R_0 x'$

$\langle x, y \rangle \, R_v \, \langle x', y' \rangle$ iff
$x = x'$ and $y R_1 y'$



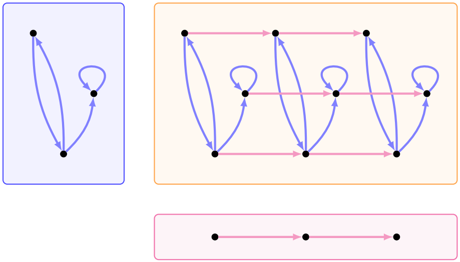


RDf$_2$ $\sim$ **Logic_of** $\{ \mathfrak{F}_0 * \mathfrak{F}_1 \mid \mathfrak{F}_0, \mathfrak{F}_1$ are equivalence relations$\}$

↑

in the language with $\Diamond_0$, $\Diamond_1$ and $\Box_0$, $\Box_1$

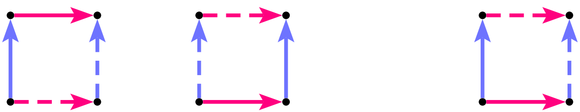
Axiomatising 2D approximations: first steps

- 

  our 2D structures contain disjoint copies of the 'component' structures, so their modally describable properties transfer to the 2D structure

- because our 2D structures are 'full rectangles', we always have the formulas

$$\Diamond_0\Diamond_1 p \leftrightarrow \Diamond_1\Diamond_0 p \quad \text{and} \quad \Diamond_0\Box_1 p \longrightarrow \Box_1\Diamond_0 p$$



commutativity and **confluence**: f-o $\forall\exists$-properties

(confluence follows from commutativity if one relation is symmetric)

Are these ENOUGH?

Axiomatising 2D approximations: known results

Gabbay–Shehtman 1998

f-o Horn formula: $\forall xy\bar{z}\,\big(\Phi(x,y,\bar{z}) \to R(x,y)\big)$ where Φ is positive

modal Horn axiom: modal formula with Horn f-o correspondent (reflexive, transitive, symmetric, . . .)

- If **Logic_of** $(\mathcal{C}_0)$ and **Logic_of** $(\mathcal{C}_1)$ are both Horn axiomatisable, then

Logic_of $(\mathcal{C}_0 * \mathcal{C}_1)$ = 'component axioms + commutativity + confluence'

Can this be generalised to universal (sub-equivalence) properties?

Is there a 'natural' non-finitely axiomatisable 2D approximation?

Axiomatising 2D approximations: 'difference' structures

- **'difference' structures:** $\langle W, \neq \rangle$
- modally describable properties: **symmetry** and

  pseudo-transitivity (universal but not Horn):

  $$\forall xyz\, \big(R(x,y) \wedge R(y,z) \rightarrow x = z \vee R(x,z)\big)$$

(we obtain equivalence by adding reflexivity)

Hampson–K 2012

- $\mathcal{C}_{\mathit{diff}}$: the class of all 'difference' structures
- $\mathcal{C}$: **any** class of 'sub-equivalence' structures

Every axiomatisation of $\textbf{Logic\_of}(\,\mathcal{C}_{\mathit{diff}} * \mathcal{C}\,)$ must contain **infinitely many** propositional variables.

Axiomatising 2D approximations: 'linear' structures

Another universal but not Horn, 'sub-equivalence' property:

weak connectedness

$$\forall xyz\, \big(R(x,y) \wedge R(x,z) \rightarrow y = z \vee R(y,z) \vee R(z,y)\big)$$

(**linearly ordered chain of clusters** if transitive and rooted)

K–Marcelino 2010

- $\mathcal{C}_{lin}$: the class of all **'linear'** (transitive and weakly connected) structures
- $\mathcal{C}_{all}$: the class of **all** structures
- $\mathcal{C}_{tr}$: the class of all **transitive** structures

Any axiomatisation of $\text{Logic\_of}(\mathcal{C}_{lin} * \mathcal{C}_{all})$ or $\text{Logic\_of}(\mathcal{C}_{lin} * \mathcal{C}_{tr})$ must contain

- **infinitely many** propositional variables
- formulas of **arbitrarily large $\Diamond_1$-depth**

Open: Is $\text{Logic\_of}(\mathcal{C}_{lin} * \mathcal{C}_{equiv})$ finitely axiomatisable?

Complexity of the satisfiability problem: some upper bounds

- *Scott 62, Mortimer 1975*

  $\mathbf{RDf_2} \sim \mathbf{Logic\_of}(\,\mathcal{C}_{equiv} * \mathcal{C}_{equiv}\,)$ is **decidable in NEXPTIME**

  has the **finite model property**

  (reducible to two-variable equality-free fragment of f-o logic)

- *Grädel–Otto–Rosen 1997, Pacholski–Swast–Tendera 1997, Pratt-Hartman 2005*

  $\mathbf{Logic\_of}(\,\mathcal{C}_{diff} * \mathcal{C}_{diff}\,)$ is still **decidable in NEXPTIME**

  NO finite model property

  (reducible to two-variable fragment of f-o logic with **counting** quantifiers)

- *Gabbay–Shehtman 1998*

  $\mathbf{Logic\_of}(\,\mathcal{C}_{all} * \mathcal{C}_{all}\,)$ is **decidable**

  has the **finite model property**

Complexity of the satisfiability problem: some lower bounds

- *Marx 1999*

  Any class of $*$-products of 'sub-equivalence' structures is NEXPTIME-**hard**

- *Göller–Jung–Lohrey 2012*

  Logic_of $(\mathcal{C}_{all} * \mathcal{C}_{all})$ is **not decidable in elementary time**

- *Gabelaia–K–Wolter–Zakharyaschev 2005*

  Transitive (but not symmetric) structures usually result in 'bad' logics

  Logic_of $(\mathcal{C}_{tr} * \mathcal{C}_{tr})$ is **undecidable**

  (same as **Logic_of**(**two commuting and confluent transitive relations**)

  or, **EqTheory_of**(**two commuting and confluent closure operators**)

- *Hampson–K 2012*

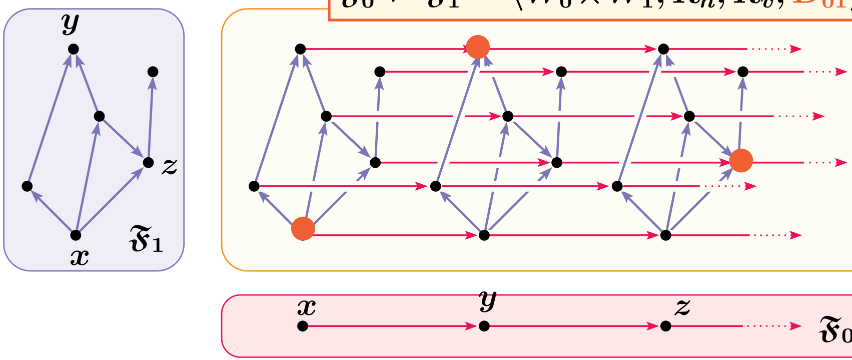
  Logic_of $(\mathcal{C}_{lin} * \mathcal{C}_{diff})$ is **undecidable**

Open: Is Logic_of $(\mathcal{C}_{tr} * \mathcal{C}_{diff})$ **decidable?**

Adding the diagonal

- We add a constant $\boxed{\delta}$ to the bimodal language with $\Diamond_0$, $\Diamond_1$, interpreted in 2D structures as $\boxed{D_{01} = \{\langle x, y\rangle \in W_0 \times W_1 \mid x = y\}}$
- **Either:** we consider 'square' products $\boxed{\mathfrak{F} *^{sq} \mathfrak{F}}$ of the **same** structure $\mathfrak{F}$
- **Or:**

$$\mathfrak{F}_0 *^{\delta} \mathfrak{F}_1 = \langle W_0 \times W_1, R_h, R_v, D_{01}\rangle$$



The **diagonal subset D_{01}** always has the following properties:

$$\forall x \in W_0,\ \forall y, y' \in W_1\ \big(\langle x, y\rangle\, ,\ \langle x, y'\rangle \in D_{01} \quad\Longrightarrow\quad y = y'\big)$$
$$\forall x, x' \in W_0,\ \forall y \in W_1\ \big(\langle x, y\rangle\, ,\ \langle x', y\rangle \in D_{01} \quad\Longrightarrow\quad x = x'\big)$$

Axiomatising 2D approximations with diagonal

- $\mathbf{RCA_2} \sim \mathbf{Logic\_of}(\mathcal{C}_{equiv} *^{sq} \mathcal{C}_{equiv})$: well-known finite axiomatisation

- *Kikot 2010*

  Every axiomatisation of $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{\delta} \mathcal{C}_{all})$ or $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{sq} \mathcal{C}_{all})$

  must contain **infinitely many** propositional variables.

- *K 2010*

  $\mathbf{RCA_2} \sim \mathbf{Logic\_of}(\mathcal{C}_{equiv} *^{sq} \mathcal{C}_{equiv})$ is **finitely axiomatisable over**

  $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{sq} \mathcal{C}_{all})$

Satisfiability problems of 2D approximations with diagonal

- *Mortimer 1975*

  $\mathbf{RCA_2} \sim \mathbf{Logic\_of}(\mathcal{C}_{equiv} *^{sq} \mathcal{C}_{equiv})$ is **decidable in NEXPTIME**

  has the **finite model property**

  (reducible to two-variable fragment of f-o logic)

- *K–Kikot 2010*

  $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{sq} \mathcal{C}_{all})$ $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{\delta} \mathcal{C}_{all})$

  $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{sq} \mathcal{C}_{tr})$ $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{\delta} \mathcal{C}_{tr})$ are all **undecidable**

  (these are all **decidable without diagonal**)

Open: Is $\mathbf{Logic\_of}(\mathcal{C}_{all} *^{\delta} \mathcal{C}_{equiv})$ decidable?

Possible future directions

- connections with other extensions of the two-variable f-o fragment
  - with equivalence closure (*Kieronski–Pratt-Hartmann–Tendera 2012*)
  - various undecidable extensions (*Grädel–Otto–Rosen 1999*)
- less interaction between the components (relativisations)

  Open: Is the logic of two commuting transitive relations decidable?

- reducts over 2D structures
  - positive
  - $(\Diamond_i, \wedge)$ (positive existential queries, description logic $\mathcal{EL}$, reflection calculus in provability logics)
  - unary negation (*ten Cate–Segoufin 2011*)
- connections with f-o guarded fragments

  Andréka–van Benthem–Németi 1998

  Marx 2001, Grädel 1999

  Bárány–ten Cate–Segoufin 2011
- other logical properties over 2D structures: interpolation, definability, . . .