Approximating the 2-variable fragment of predicate logic with propositional modal logics: a survey of recent results

Agi Kurucz

Department of Informatics
King’s College London
Propositional bimodal logics

- **bimodal formulas:**
  \[
  \varphi = p | \neg \varphi | \varphi \lor \psi | \Diamond_0 \varphi | \Diamond_1 \varphi | \Box_0 \varphi | \Box_1 \varphi
  \]

- **relational structures:**
  \[
  \mathcal{F} = \langle W, R_0, R_1 \rangle
  \]

- **models:**
  \[
  \mathcal{M} = \langle \mathcal{F}, \nu \rangle, \text{ where } \nu : \text{Variables} \rightarrow 2^W
  \]

- \((\mathcal{M}, w) \models p\) iff \(w \in \nu(p)\)
- \((\mathcal{M}, w) \models \neg \varphi\) iff \((\mathcal{M}, w) \not\models \varphi\)
- \((\mathcal{M}, w) \models \varphi \lor \psi\) iff \((\mathcal{M}, w) \models \varphi\) or \((\mathcal{M}, w) \models \psi\)
- \((\mathcal{M}, w) \models \Diamond_i \varphi\) iff \(\exists v \ (w R_i v \text{ and } (\mathcal{M}, v) \models \varphi)\)
- \((\mathcal{M}, w) \models \Box_i \varphi\) iff \(\forall v \ (\text{if } w R_i v \text{ then } (\mathcal{M}, v) \models \varphi)\)

- **validity:**
  \[
  \mathcal{F} \models \varphi\quad \text{iff}\quad (\mathcal{M}, w) \models \varphi, \text{ for all } \mathcal{M} = \langle \mathcal{F}, \nu \rangle \text{ and } w \in W
  \]
Modal logics vs. Boolean Algebras with Operators

- modal operator $\Diamond_i$
- formula $\varphi$
- relational structure $\mathcal{F}$
- $\mathcal{F} \models \varphi$
- Logic of $\mathcal{C} = \{ \varphi : \mathcal{F} \models \varphi, \forall \mathcal{F} \in \mathcal{C} \}$
  - finitely axiomatisable
  - decidable
  - finite model property
- unary function symbol $f^{\Diamond_i}$
- term $t_\varphi$
- full complex algebra $Cm(\mathcal{F})$ of $\mathcal{F}$
- $Cm(\mathcal{F}) \models (t_\varphi = 1)$
- EqTheory of Var$\{ Cm(\mathcal{F}) : \mathcal{F} \in \mathcal{C} \}$
  - finitely based
  - decidable
  - finite algebra property
2D cylindric set algebras

Subalgebras of the full complex algebra of 2D ‘square-structures’ of the form

\[ \langle W \times W, \equiv_0, \equiv_1, D_{01} \rangle \]

where, for all \( \langle x, y \rangle, \langle x', y' \rangle \in W \times W \),

\[ \langle x, y \rangle \equiv_0 \langle x', y' \rangle \iff y = y' \]
\[ \langle x, y \rangle \equiv_1 \langle x', y' \rangle \iff x = x' \]
\[ D_{01} = \{ \langle x, x \rangle \mid x \in W \} \]

\[ \text{RCA}_2 = \text{Var}\{ Cm \langle W \times W, \equiv_0, \equiv_1, D_{01} \rangle \mid W \text{ is a nonempty set} \} \]
Modal logic (or RCA$_2$) vs. two-variable predicate logic

multimodal formula $\varphi$ $\leadsto$ f-o formula $\varphi^*$ with $\leq 2$ variables

- $p \mapsto P(x,y)$
- $\neg \varphi \mapsto \neg \varphi^*$
- $\varphi \lor \psi \mapsto \varphi^* \lor \psi^*$
- $\Diamond 0\varphi \mapsto \exists x \varphi^*$
- $\Box 0\varphi \mapsto \forall x \varphi^*$
- $\Diamond 1\varphi \mapsto \exists y \varphi^*$
- $\Box 1\varphi \mapsto \forall y \varphi^*$
- $\delta \mapsto x = y$

$\varphi$ is valid in ‘square-structures’ $\iff$ $\varphi^*$ is f-o valid
2D propositional approximations of $\text{RCA}_2$

We classify the properties of $\text{RCA}_2$:

- **Group (0):** BAO properties
  
- **Group (1):** additional properties of the *individual* cylindrifications
  
- **Group (2):** ‘*dimension-connecting*’ properties due to the 2D structure

What’s next? We try to keep (0) and (2), but relax (1)

$\rightsquigarrow$ 2D structures with possibly ‘*sub-equivalence*’ relations
Diagonal-free case: 2D relational structures


For $\mathcal{F}_0 = \langle W_0, R_0 \rangle$ and $\mathcal{F}_1 = \langle W_1, R_1 \rangle$, the $*$-product structure is $\mathcal{F}_0 * \mathcal{F}_1 = \langle W_0 \times W_1, R_h, R_v \rangle$, where

- $\langle x, y \rangle R_h \langle x', y' \rangle$ iff $y = y'$ and $x R_0 x'$
- $\langle x, y \rangle R_v \langle x', y' \rangle$ iff $x = x'$ and $y R_1 y'$

$\mathcal{F}_0 * \mathcal{F}_1$ $\sim$ Logic of $\{ \mathcal{F}_0 * \mathcal{F}_1 \mid \mathcal{F}_0, \mathcal{F}_1 \text{ are equivalence relations} \}$

↑
in the language with $\Diamond_0, \Diamond_1$ and $\Box_0, \Box_1$
Axiomatising 2D approximations: first steps

- our 2D structures contain disjoint copies of the ‘component’ structures, so their modally describable properties transfer to the 2D structure

- because our 2D structures are ‘full rectangles’, we always have the formulas

\[ \Diamond_0 \Diamond_1 p \leftrightarrow \Diamond_1 \Diamond_0 p \]  
\[ \Diamond_0 \Box_1 p \rightarrow \Box_1 \Diamond_0 p \]

**commutativity** and **confluence**: f-o ∀∃-properties

(confluence follows from commutativity if one relation is symmetric)

Are these ENOUGH?
Axiomatising 2D approximations: known results

Gabbay–Shehtman 1998

f-o Horn formula: \[ \forall x y z (\Phi(x, y, z) \rightarrow R(x, y)) \] where \( \Phi \) is positive

modal Horn axiom: modal formula with Horn f-o correspondent (reflexive, transitive, symmetric, ...)

- If \( \text{Logic of } (C_0) \) and \( \text{Logic of } (C_1) \) are both Horn axiomatisable, then \[ \text{Logic of } (C_0 \ast C_1) = \text{‘component axioms + commutativity + confluence’} \]

Can this be generalised to universal (sub-equivalence) properties?

Is there a ‘natural’ non-finitely axiomatisable 2D approximation?
Axiomatising 2D approximations: ‘difference’ structures

- ‘difference’ structures: \( \langle W, \neq \rangle \)

- modally describable properties: symmetry and pseudo-transitivity (universal but not Horn):

\[
\forall x y z \ (R(x, y) \land R(y, z) \rightarrow x = z \lor R(x, z))
\]

(we obtain equivalence by adding reflexivity)

Hampson–K 2012

- \( C_{\text{diff}} \): the class of all ‘difference’ structures
- \( C \): any class of ‘sub-equivalence’ structures

Every axiomatisation of \( \text{Logic}_\text{of}(C_{\text{diff}} \ast C) \) must contain infinitely many propositional variables.
Axiomatising 2D approximations: ‘linear’ structures

Another universal but not Horn, ‘sub-equivalence’ property:

weak connectedness

\[ \forall xyz \ (R(x, y) \land R(x, z) \rightarrow y = z \lor R(y, z) \lor R(z, y)) \]

(linearly ordered chain of clusters if transitive and rooted)

K–Marcelino 2010

- \( C_{\text{lin}} \): the class of all ‘linear’ (transitive and weakly connected) structures
- \( C_{\text{all}} \): the class of all structures
- \( C_{\text{tr}} \): the class of all transitive structures

Any axiomatisation of \( \text{Logic}_o f (C_{\text{lin}} \ast C_{\text{all}}) \) or \( \text{Logic}_o f (C_{\text{lin}} \ast C_{\text{tr}}) \) must contain

- infinitely many propositional variables
- formulas of arbitrarily large \( \Diamond_1 \)-depth

Open: Is \( \text{Logic}_o f (C_{\text{lin}} \ast C_{\text{equiv}}) \) finitely axiomatisable?
Complexity of the satisfiability problem: some upper bounds

- **Scott 62, Mortimer 1975**
  
  $\text{RDF}_2 \sim \text{Logic}_\text{of}(C_{\text{equiv}} \ast C_{\text{equiv}})$ is decidable in NEXPTIME

  has the finite model property

  (reducible to two-variable equality-free fragment of f-o logic)

  
  $\text{Logic}_\text{of}(C_{\text{diff}} \ast C_{\text{diff}})$ is still decidable in NEXPTIME

  NO finite model property

  (reducible to two-variable fragment of f-o logic with counting quantifiers)

- **Gabbay–Shehtman 1998**
  
  $\text{Logic}_\text{of}(C_{\text{all}} \ast C_{\text{all}})$ is decidable

  has the finite model property
Complexity of the satisfiability problem: some lower bounds

- Marx 1999
  Any class of $*$-products of ‘sub-equivalence’ structures is \textsc{NEXPTIME-hard}.

- Goller–Jung–Lohrey 2012
  $\text{Logic}_{\text{of}} \left( C_{\text{all}} \ast C_{\text{all}} \right)$ is not decidable in elementary time.

  Transitive (but not symmetric) structures usually result in ‘bad’ logics.
  $\text{Logic}_{\text{of}} \left( C_{\text{tr}} \ast C_{\text{tr}} \right)$ is undecidable.
  (same as $\text{Logic}_{\text{of}}$\text{(two commuting and confluent transitive relations)}
  or, $\text{EqTheory}_{\text{of}}$\text{(two commuting and confluent closure operators)}

- Hampson–K 2012
  $\text{Logic}_{\text{of}} \left( C_{\text{lin}} \ast C_{\text{diff}} \right)$ is undecidable.

Open: Is $\text{Logic}_{\text{of}} \left( C_{\text{tr}} \ast C_{\text{diff}} \right)$ decidable?
Adding the diagonal

- We add a constant $\delta$ to the bimodal language with $\Diamond_0$, $\Diamond_1$, interpreted in 2D structures as
  \[ D_{01} = \{ \langle x, y \rangle \in W_0 \times W_1 \mid x = y \} \]

- Either: we consider ‘square’ products $\mathcal{F} \ast_{sq} \mathcal{F}$ of the same structure $\mathcal{F}$

- Or:

The diagonal subset $D_{01}$ always has the following properties:

- $\forall x \in W_0, \forall y, y' \in W_1 \ (\langle x, y \rangle, \langle x, y' \rangle \in D_{01} \implies y = y')$
- $\forall x, x' \in W_0, \forall y \in W_1 \ (\langle x, y \rangle, \langle x', y \rangle \in D_{01} \implies x = x')$
Axiomatising 2D approximations with diagonal

- $\text{RCA}_2 \sim \text{Logic}_\text{of}(C_{\text{equiv}} \ast_{\text{sq}} C_{\text{equiv}})$: well-known finite axiomatisation

- Kikot 2010

Every axiomatisation of $\text{Logic}_\text{of}(C_{\text{all}} \ast_\delta C_{\text{all}})$ or $\text{Logic}_\text{of}(C_{\text{all}} \ast_{\text{sq}} C_{\text{all}})$ must contain infinitely many propositional variables.

- K 2010

$\text{RCA}_2 \sim \text{Logic}_\text{of}(C_{\text{equiv}} \ast_{\text{sq}} C_{\text{equiv}})$ is finitely axiomatisable over $\text{Logic}_\text{of}(C_{\text{all}} \ast_{\text{sq}} C_{\text{all}})$
Satisfiability problems of 2D approximations with diagonal

- Mortimer 1975
  \[ \text{RCA}_2 \sim \text{Logic}_{\text{of}}(C_{\text{equiv}} \ast \text{sq} \ C_{\text{equiv}}) \]
  is \textit{decidable in NEXPTIME}
  has the \textit{finite model property}
  (reducible to two-variable fragment of f-o logic)

- K–Kikot 2010
  \[ \text{Logic}_{\text{of}}(C_{\text{all}} \ast \text{sq} \ C_{\text{all}}) \]
  \[ \text{Logic}_{\text{of}}(C_{\text{all}} \ast \delta \ C_{\text{all}}) \]
  \[ \text{Logic}_{\text{of}}(C_{\text{all}} \ast \text{sq} \ C_{\text{tr}}) \]
  \[ \text{Logic}_{\text{of}}(C_{\text{all}} \ast \delta \ C_{\text{tr}}) \]
  are all \textit{undecidable}
  (these are all \textit{decidable without diagonal})

Open: Is \text{Logic}_{\text{of}}(C_{\text{all}} \ast \delta \ C_{\text{equiv}}) \textit{decidable?}
Possible future directions

- connections with other extensions of the two-variable f-o fragment
  - with equivalence closure (Kieronski–Pratt-Hartmann–Tendera 2012)
  - various undecidable extensions (Grädel–Otto–Rosen 1999)

- less interaction between the components (relativisations)

  Open: Is the logic of two commuting transitive relations decidable?

- reducts over 2D structures
  - positive
  - $\left(\diamond_i, \land\right)$ (positive existential queries, description logic $\mathcal{EL}$, reflection calculus in provability logics)

- unary negation (ten Cate–Segoufin 2011)

- connections with f-o guarded fragments
  
  Andréka–van Benthem–Németi 1998
  Marx 2001, Grädel 1999
  Bárány–ten Cate–Segoufin 2011

- other logical properties over 2D structures: interpolation, definability, . . .