Strongly representable atom structures

Mohamed Khaled and Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

Outline

Outline

Boolean algebra

A ***boolean algebra*** is an algebraic structure $\langle A, +, \cdot, -, 0, 1\rangle$, in which $+, \cdot$ are two binary operation on A, while $-$ is a unary operation on A, and 0, 1 are distinguished elements of A, such that the following postulates hold for arbitrary $x, y, z \in A$:

$$
\begin{array}{lcc}
B_0 & x + y = y + x, & x \cdot y = y \cdot x, \\
B_1 & x + (y \cdot z) = (x + y) \cdot (x + z), & x \cdot (y + z) = x \cdot y + x \cdot z, \\
B_2 & x + 0 = x, & x \cdot 1 = x, \\
B_3 & x + -x = 1, & x \cdot -x = 0.
\end{array}
$$

Boolean algebra with operators

Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1 \rangle$ be a boolean algebra.

- An *operator* Ω with arity or rank $rk(\Omega) = n < \omega$ on $\mathfrak{B}$ is a function $\Omega :^n B \to B$ such that:

Boolean algebra with operators

Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1\rangle$ be a boolean algebra.

- An *operator* Ω with arity or rank $rk(\Omega) = n < \omega$ on $\mathfrak{B}$ is a function $\Omega :^n B \to B$ such that:
    1. Ω is normal: for all $b_0, \cdots, b_{n-1} \in \mathfrak{B}$ and $i < n$, if $b_i = 0$ then $\Omega(b_0, \cdots, b_{n-1}) = 0$,

Boolean algebra with operators

Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1 \rangle$ be a boolean algebra.

- An *operator* Ω with arity or rank $rk(\Omega) = n < \omega$ on $\mathfrak{B}$ is a function $\Omega :^n B \to B$ such that:
  1. Ω is normal: for all $b_0, \cdots, b_{n-1} \in \mathfrak{B}$ and $i < n$, if $b_i = 0$ then $\Omega(b_0, \cdots, b_{n-1}) = 0$,
  2. Ω is additive: for any $b_0, \cdots, b_{n-1}, b, b' \in \mathfrak{B}$ and $i < n$, we have

$$\begin{aligned}\Omega(b_0 &, \cdots, b_{i-1}, (b + b'), b_{i+1}, \cdots, b_{n-1}) \\ &= \Omega(b_0, \cdots, b_{i-1}, \cdots, b, b_{i+1}, \cdots, b_{n-1}) \\ &\quad + \Omega(b_0, \cdots, b_{i-1}, b', b_{i+1} \cdots, b_{n-1}).\end{aligned}$$

Boolean algebra with operators

Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1\rangle$ be a boolean algebra.

- An *operator* Ω with arity or rank $rk(\Omega) = n < \omega$ on $\mathfrak{B}$ is a function $\Omega :^n B \to B$ such that:
  1. Ω is normal: for all $b_0, \cdots, b_{n-1} \in \mathfrak{B}$ and $i < n$, if $b_i = 0$ then $\Omega(b_0, \cdots, b_{n-1}) = 0$,
  2. Ω is additive: for any $b_0, \cdots, b_{n-1}, b, b' \in \mathfrak{B}$ and $i < n$, we have

$$\begin{aligned}\Omega(b_0 \quad, &\quad \cdots, b_{i-1}, (b + b'), b_{i+1}, \cdots, b_{n-1}) \\ = &\quad \Omega(b_0, \cdots, b_{i-1}, \cdots, b, b_{i+1}, \cdots, b_{n-1}) \\ &\quad + \Omega(b_0, \cdots, b_{i-1}, b', b_{i+1} \cdots, b_{n-1}).\end{aligned}$$

We may call Ω an n-ary operator.

Boolean algebra with operators

Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1\rangle$ be a boolean algebra.

- A *boolean algebra with operators BAO* is an algebra $\langle B, +, \cdot, -, 0, 1, \Omega_\lambda\rangle_{\lambda\in\Lambda}$, where $\langle B, +, \cdot, -, 0, 1\rangle$ is a boolean algebra, Λ is a set (perhaps uncountable), and Ω_λ $(\lambda \in \Lambda)$ are operators on $\mathfrak{B}$.

Cylindric algebra

Let α be an ordinal. By a ***cylindric algebra*** of dimension α we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$$

where 0, 1 and $d_{\kappa\lambda}$ are distinguished elements of A (for all $\kappa, \lambda < \alpha$), $+$ and $\cdot$ are two binary operations on A, $-$ and c_κ are unary operations on A (for all $\kappa < \alpha$), and such that the following postulates are satisfies for any $x, y \in A$ and any $\kappa, \lambda, \mu < \alpha$:

- C_0. $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra,
- C_1. $c_i 0 = 0$,
- C_2. $x \leq c_i x$ (i.e, $x + c_i x = c_i x$),
- C_3. $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,

Cylindric algebra

Let α be an ordinal. By a ***cylindric algebra*** of dimension α we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$$

where 0, 1 and $d_{\kappa\lambda}$ are distinguished elements of A (for all $\kappa, \lambda < \alpha$), $+$ and $\cdot$ are two binary operations on A, $-$ and c_κ are unary operations on A (for all $\kappa < \alpha$), and such that the following postulates are satisfies for any $x, y \in A$ and any $\kappa, \lambda, \mu < \alpha$:

- C_4. $c_i c_j x = c_j c_i x$,
- C_5. $d_{ii} = 1$,
- C_6. if $i \neq j, \mu$, then $d_{j\mu} = c_i(d_{ji} \cdot d_{i\mu})$,
- C_7. if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

Diagonal-free Cylindric algebra

Let α be an ordinal. A ***diagonal-free cylindric algebra*** of dimension α, is a structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i \rangle_{i<\alpha}$$

satisfying the following for every $x, y \in A$, and $i, j < \alpha$:

Df_0. The reduct $\langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra,

Df_1. $c_i 0 = 0$,

Df_2. $x \leq c_i x$,

Df_3. $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,

Df_4. $c_i c_j x = c_j c_i x$.

Substitution algebra

Let α be an ordinal. By a ***substitution algebra*** of dimension α, briefly an SC_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s^i_j \rangle_{i,j<\alpha}$$

where$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra and c_i, s^i_j are unary operations on A $(i, j < \alpha)$ satisfying the following equations for all $x \in A$, and all $i, j, \kappa, l < \alpha$:

S_0. $c_i 0 = 0$, $x \leq c_i x$ (i.e, $x + c_i x = c_i x$),
$c_i(x \cdot c_i y) = c_i x \cdot c_i y$, and $c_i c_j x = c_j c_i x$,

S_1. $s^i_i x = x$,

S_2. s^i_j are boolean endomorphisms,

S_3. $s^i_j c_i x = c_i x$,

S_4. $c_i s^i_j x = s^i_j x$, whenever $i \neq j$,

Substitution algebra

Let α be an ordinal. By a ***substitution algebra*** of dimension α, briefly an SC_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i \rangle_{i,j<\alpha}$$

where$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra and c_i, s_j^i are unary operations on A $(i, j < \alpha)$ satisfying the following equations for all $x \in A$, and all $i, j, \kappa, l < \alpha$:

S_5. $s_j^i c_\kappa x = c_\kappa s_j^i x$, whenever $\kappa \notin \{i, j\}$,

S_6. $c_i s_i^j x = c_j s_j^i x$,

S_7. $s_j^i s_\kappa^l x = s_\kappa^l s_j^i x$, whenever $|\{i, j, \kappa, l\}| = 4$,

S_8. $s_i^l s_l^j x = s_i^l s_i^j x$.

Quasi polyadic algebra

Let α be an ordinal. By a ***quasi polyadic algebra*** of dimension α, briefly a QA_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s^i_j, s_{ij}\rangle_{i,j<\alpha}$$

where $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$ is a boolean algebra, c_i, s^i_j, and s_{ij} are unary operations on A (for $i, j < \alpha$), and the following postulates are satisfied for all $x \in A$, and all $i, j, \kappa < \alpha$:

- Q_0. $s^i_i = s_{ii} = Id$, and $s_{ij} = s_{ji}$,
- Q_1. $x \leq c_i x$ (i.e, $x + c_i x = c_i x$),
- Q_2. $c_i(x + y) = c_i x + c_i y$,
- Q_3. $s^i_j c_i x = c_i x$,
- Q_4. if $i \neq j$, then $c_i s^i_j x = s^i_j x$,

Quasi polyadic algebra

Let α be an ordinal. By a ***quasi polyadic algebra*** of dimension α, briefly a QA_α, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, s^i_j, s_{ij} \rangle_{i,j<\alpha}$$

where $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra, c_i, s^i_j, and s_{ij} are unary operations on A (for $i, j < \alpha$), and the following postulates are satisfied for all $x \in A$, and all $i, j, \kappa < \alpha$:

Q_5. if $\kappa \notin \{i, j\}$, then $s^i_j c_\kappa x = c_\kappa s^i_j x$,

Q_6. s^i_j, and s_{ij} are boolean endomorphisms,

Q_7. $s_{ij} s_{ij} x = x$,

Q_8. if $|\{i, j, \kappa\}| = 3$, then $s_{ij} s_{i\kappa} x = s_{j\kappa} s_{ij} x$,

Q_9. $s_{ij} s^i_j x = s^j_i x$.

Quasi polyadic equality algebra

Let α be an ordinal. By a ***quasi polyadic equality algebra*** of dimension α, briefly a QEA_α, we mean an algebra

$$\mathfrak{B} = \langle \mathfrak{A}, d_{ij} \rangle_{i,j<\alpha}$$

where $\mathfrak{A}$ is a QA_α, d_{ij} is a constant (for $i, j < \alpha$) and the following postulates are satisfied for all $i, j, \kappa < \alpha$:

QE_0. $d_{ii} = 1$,

QE_1. $s^i_j d_{ij} = 1$,

QE_2. $x \cdot d_{ij} \leq s^i_j x$.

The action of the non-boolean operators in a completely additive atomic *BAO* is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra.

The action of the non-boolean operators in a completely additive atomic *BAO* is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra.

Atom Structure

Let $\mathcal{A} = \langle A, +, -, 0, 1, \Omega_i : i \in I \rangle$ be an atomic boolean algebra with operators $\Omega_i : i \in I$. Let the rank of Ω_i be ρ_i. The *atom structure* $At\mathcal{A}$ of $\mathcal{A}$ is a relational structure

$$\langle At\mathcal{A}, R_{\Omega_i} : i \in I \rangle$$

where $At\mathcal{A}$ is the set of atoms of $\mathcal{A}$ as before, and R_{Ω_i} is a $(\rho(i) + 1)$-ary relation over $At\mathcal{A}$ defined by

$$R_{\Omega_i}(a_0, \cdots, a_{\rho(i)}) \Longleftrightarrow \Omega_i(a_1, \cdots, a_{\rho(i)}) \geq a_0.$$

Complex algebra

Conversely, if we are given an arbitrary structure $\mathcal{S} = \langle S, r_i : i \in I \rangle$ where r_i is a $(\rho(i)+1)$-ary relation over S, we can define its *complex algebra*

$$\mathfrak{Cm}(\mathcal{S}) = \langle \wp(S), \cup, \setminus, \phi, S, \Omega_i \rangle_{i \in I},$$

where $\wp(S)$ is the power set of S, and Ω_i is the $\rho(i)$-ary operator defined by

$$\Omega_i(X_1, \cdots, X_{\rho(i)}) = \{s \in S : \exists \quad s_1 \in X_1 \cdots \exists s_{\rho(i)} \in X_{\rho(i)}, r_i(s, s_1, \cdots, s_{\rho(i)})\},$$

for each $X_1, \cdots, X_{\rho(i)} \in \wp(S)$.

Atom structure of diagonal free-type algebra is

$$\mathcal{S} = \langle S, R_{c_i} : i < n \rangle,$$

where the R_{c_i} is binary relation on S.

Atom structure of cylindric-type algebra is

$$\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}} : i, j < n \rangle,$$

where the $R_{d_{ij}}$, R_{c_i} are unary and binary relations on S. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ is an atom structure of diagonal free-type.

Atom structure of substitution-type algebra is

$$\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j < n \rangle,$$

where the $R_{d_{ij}}$, $R_{s^i_j}$ are unary and binary relations on S, respectively. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ is an atom structure of diagonal free-type.

Atom structure of quasi polyadic-type algebra is

$$\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j}, R_{s_{ij}} : i, j < n \rangle,$$

where the R_{c_i}, $R_{s^i_j}$ and $R_{s_{ij}}$ are binary relations on S. The reducts $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ and $\mathfrak{Rd}_{Sc}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j < n \rangle$ are atom structures of diagonal free and substitution types, respectively.

Atom structure of quasi polyadic equality-type algebra is

$$\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}}, R_{s^i_j}, R_{s_{ij}} : i, j < n\rangle,$$

where the $R_{d_{ij}}$ is unary relation on S, and R_{c_i}, $R_{s^i_j}$ and $R_{s_{ij}}$ are binary relations on S.

- The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i \in I\rangle$ is an atom structure of diagonal free-type.
- The reduct $\mathfrak{Rd}_{ca}\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}} : i, j \in I\rangle$ is an atom structure of cylindric-type.
- The reduct $\mathfrak{Rd}_{Sc}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j \in I\rangle$ is an atom structure of substitution-type.
- The reduct $\mathfrak{Rd}_{qa}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j}, R_{s_{ij}} : i, j \in I\rangle$ is an atom structure of quasi polyadic-type.

Definition

An algebra is said to be representable if and only if it is isomorphic to a subalgebra of a direct product of set algebras of the same type.

Definition

An algebra is said to be representable if and only if it is isomorphic to a subalgebra of a direct product of set algebras of the same type.

Definition

Let $\mathcal{S}$ be an n-dimensional algebra atom structure. $\mathcal{S}$ is *strongly representable* if every atomic n-dimensional algebra $\mathcal{A}$ with $At\mathcal{A} = \mathcal{S}$ is representable. We write $SDfS_n$, SCS_n, $SSCS_n$, SQS_n and $SQES_n$ for the classes of strongly representable (n-dimensional) diagonal free, cylindric, substitution, quasi polyadic and quasi polyadic equality algebra atom structures, respectively.

Note that for any n-dimensional algebra $\mathcal{A}$ and atom structure $\mathcal{S}$, if $At\mathcal{A} = \mathcal{S}$ then $\mathcal{A}$ embeds into $\mathfrak{Cm}\mathcal{S}$, and hence $\mathcal{S}$ is strongly representable iff $\mathfrak{Cm}\mathcal{S}$ is representable.

Hirsch and Hodkinson proved that for finite $n \geq 3$, the class of strongly representable cylindric-type atom structures of dimension n is not definable by any set of first-order sentences: it is not elementary class.

Their method depends on that RCA_n is a variety, an atomic algebra $\mathfrak{A}$ will be in RCA_n if all the equations defining RCA_n are valid in $\mathfrak{A}$. From the point of view of $At\mathfrak{A}$, each equation corresponds to a certain universal monadic second-order statement, where the universal quantifiers are restricted to ranging over the sets of atoms that are defined by elements of $\mathfrak{A}$. Such a statement will fail in $\mathfrak{A}$ if $At\mathfrak{A}$ can be partitioned into finitely many $\mathfrak{A}$-definable sets with certain properties - they call this a bad partition.

This idea can be used to show that RCA_n (for $n \geq 3$) is not finitely axiomatizable, by finding a sequence of atom structures, each having some sets that form a bad partition, but with the minimal number of sets in a bad partition increasing as we go along the sequence. This can yield algebras not in RCA_n but with an ultraproduct that is in RCA_n.

In this article we extend the result of Hirsch and Hodkinson to any class of strongly representable atom structure having signature between the diagonal free atom structures and the quasi polyadic equality atom structures.

Outline

In this section, by a graph we will mean a pair $\Gamma = (G, E)$, where $G \neq \phi$ and $E \subseteq G \times G$ is a reflexive and symmetric binary relation on G. We will often use the same notation for Γ and for its set of nodes (G above). A pair $(x, y) \in E$ will be called an edge of Γ. See [5] for basic information (and a lot more) about graphs.

In this section, by a graph we will mean a pair $\Gamma = (G, E)$, where $G \neq \phi$ and $E \subseteq G \times G$ is a reflexive and symmetric binary relation on G. We will often use the same notation for Γ and for its set of nodes (G above). A pair $(x, y) \in E$ will be called an edge of Γ. See [5] for basic information (and a lot more) about graphs.

Definition

Let $\Gamma = (G, E)$ be a graph.

1. A set $X \subset G$ is said to be *independent* if $E \cap (X \times X) = \phi$.
2. The *chromatic number* $\chi(\Gamma)$ of Γ is the smallest $\kappa < \omega$ such that G can be partitioned into κ independent sets, and ∞ if there is no such κ.

Definition

- For an equivalence relation $\sim$ on a set X, and $Y \subseteq X$, we write $\sim\restriction Y$ for $\sim \cap (Y \times Y)$. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i)$, $K(j)$ are both undefined, or they are both defined and are equal.
- For any two relations $\sim$ and $\approx$. The composition of $\sim$ and $\approx$ is the set

$$\sim \circ \approx = \{(a, b) : \exists c(a \sim c \wedge c \approx b)\}.$$

Definition

- For an equivalence relation $\sim$ on a set X, and $Y \subseteq X$, we write $\sim\restriction Y$ for $\sim \cap (Y \times Y)$. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i)$, $K(j)$ are both undefined, or they are both defined and are equal.
- For any two relations $\sim$ and $\approx$. The composition of $\sim$ and $\approx$ is the set

$$\sim \circ \approx = \{(a, b) : \exists c(a \sim c \land c \approx b)\}.$$

Definition

Let Γ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n \rangle$ as follows:

1. H is the set of all pairs $(K, \sim)$ where $K : n \to \Gamma \times n$ is a partial map and $\sim$ is an equivalent relation on n satisfying the following conditions
   a. If $|n/\sim| = n$, then $dom(K) = n$ and $rng(K)$ is not independent subset of n.
   b. If $|n/\sim| = n-1$, then K is defined only on the unique $\sim$ class $\{i, j\}$ say of size 2 and $K(i) = K(j)$.
   c. If $|n/\sim| \leq n-2$, then K is nowhere defined.

Definition

Let Γ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n\rangle$ as follows:

2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.
3. $(K, \sim) \equiv_i (K', \sim')$ iff $K(i) = K'(i)$ and $\sim\restriction (n \setminus \{i\}) = \sim' \restriction (n \setminus \{i\})$.
4. $(K, \sim) \equiv_{ij} (K', \sim')$ iff $K(i) = K'(j)$, $K(j) = K'(i)$, and $K(\kappa) = K'(\kappa) (\forall \kappa \in n \setminus \{i, j\})$ and if $i \sim j$ then $\sim = \sim'$, if not, then $\sim' = \sim \circ [i, j]$.

Definition

Let Γ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n\rangle$ as follows:

2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.
3. $(K, \sim) \equiv_i (K', \sim')$ iff $K(i) = K'(i)$ and $\sim\restriction (n \setminus \{i\}) = \sim'\restriction (n \setminus \{i\})$.
4. $(K, \sim) \equiv_{ij} (K', \sim')$ iff $K(i) = K'(j)$, $K(j) = K'(i)$, and $K(\kappa) = K'(\kappa) (\forall \kappa \in n \setminus \{i, j\})$ and if $i \sim j$ then $\sim = \sim'$, if not, then $\sim' = \sim \circ [i, j]$.

Definition

Let Γ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} \colon i, j < n \rangle$ as follows:

2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.
3. $(K, \sim) \equiv_i (K', \sim')$ iff $K(i) = K'(i)$ and $\sim\restriction (n \setminus \{i\}) = \sim' \restriction (n \setminus \{i\})$.
4. $(K, \sim) \equiv_{ij} (K', \sim')$ iff $K(i) = K'(j)$, $K(j) = K'(i)$, and $K(\kappa) = K'(\kappa) (\forall \kappa \in n \setminus \{i, j\})$ and if $i \sim j$ then $\sim = \sim'$, if not, then $\sim' = \sim \circ [i, j]$.

It may help to think of $K(i)$ as assigning the nodes $K(i)$ of $\Gamma \times n$ not to i but to the set $n \setminus \{i\}$, so long as its elements are pairwise non-equivalent via $\sim$.

For a set X, $\mathcal{B}(X)$ denotes the boolean algebra $\langle \wp(X), \cup, \setminus \rangle$. We write $a \cap b$ for $-(-a \cup -b)$.

For a set X, $\mathcal{B}(X)$ denotes the boolean algebra $\langle\wp(X), \cup, \setminus\rangle$. We write $a \cap b$ for $-(-a \cup -b)$.

Definition

Let $\mathfrak{B}(\Gamma) = \langle \mathcal{B}(\eta(\Gamma)), c_i, s_j^i, s_{ij}, d_{ij}\rangle_{i,j<n}$ be the algebra, with extra non-Boolean operations defined as follows:

$$d_{ij} = D_{ij},$$
$$c_i X = \{c : \exists a \in X, a \equiv_i c\},$$
$$s_{ij} X = \{c : \exists a \in X, a \equiv_{ij} c\},$$
$$s_j^i X = \begin{cases} c_i(X \cap D_{ij}), & \text{if } i \neq j, \\ X, & \text{if } i = j. \end{cases} \text{For all } X \subseteq \eta(\Gamma).$$

For a set X, $\mathcal{B}(X)$ denotes the boolean algebra $\langle\wp(X), \cup, \setminus\rangle$. We write $a \cap b$ for $-(-a \cup -b)$.

Definition

Let $\mathfrak{B}(\Gamma) = \langle \mathcal{B}(\eta(\Gamma)), c_i, s^i_j, s_{ij}, d_{ij}\rangle_{i,j<n}$ be the algebra, with extra non-Boolean operations defined as follows:

$$d_{ij} = D_{ij},$$
$$c_i X = \{c : \exists a \in X, a \equiv_i c\},$$
$$s_{ij} X = \{c : \exists a \in X, a \equiv_{ij} c\},$$
$$s^i_j X = \begin{cases} c_i(X \cap D_{ij}), & \text{if } i \neq j, \\ X, & \text{if } i = j. \end{cases} \text{ For all } X \subseteq \eta(\Gamma).$$

For any $\tau \in \{\pi \in n^n : \pi \text{ is a bijection}\}$, and any $(K, \sim) \in \eta(\Gamma)$. We define $\tau(K, \sim) = (K \circ \tau, \sim \circ\tau)$.

The proof of the following two Lemmas is straightforward.

The proof of the following two Lemmas is straightforward.

Lemma

For any $\tau \in \{\pi \in n^n : \pi \textit{ is a bijection}\}$, *and any* $(K, \sim) \in \eta(\Gamma)$. $\tau(K, \sim) \in \eta(\Gamma)$.

Lemma

For any $(K, \sim)$, $(K', \sim')$, *and* $(K'', \sim'') \in \eta(\Gamma)$, *and* $i, j \in n$*:*

Lemma

For any $(K,\sim)$, $(K',\sim')$, *and* $(K'',\sim'')\in\eta(\Gamma)$, *and* $i,j\in n$*:*

1. $(K,\sim)\equiv_{ii}(K',\sim')\iff(K,\sim)=(K',\sim')$.

Lemma

For any $(K, \sim)$, $(K', \sim')$, *and* $(K'', \sim'') \in \eta(\Gamma)$, *and* $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim')$.
2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim')$.

Lemma

For any $(K, \sim)$, $(K', \sim')$, *and* $(K'', \sim'') \in \eta(\Gamma)$, *and* $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim')$.
2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim')$.
3. *If* $(K, \sim) \equiv_{ij} (K', \sim')$, *and* $(K, \sim) \equiv_{ij} (K'', \sim'')$, *then* $(K', \sim') = (K'', \sim'')$.

Lemma

For any $(K, \sim)$, $(K', \sim')$, *and* $(K'', \sim'') \in \eta(\Gamma)$, *and* $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim')$.
2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim')$.
3. *If* $(K, \sim) \equiv_{ij} (K', \sim')$, *and* $(K, \sim) \equiv_{ij} (K'', \sim'')$, *then* $(K', \sim') = (K'', \sim'')$.
4. *If* $(K, \sim) \in D_{ij}$, *then* $(K, \sim) \equiv_i (K', \sim') \iff \exists (K_1, \sim_1) \in \eta(\Gamma) : (K, \sim) \equiv_j (K_1, \sim_1) \wedge (K', \sim') \equiv_{ij} (K_1, \sim_1)$.

Lemma

For any $(K, \sim)$, $(K', \sim')$, and $(K'', \sim'') \in \eta(\Gamma)$, and $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim')$.
2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim')$.
3. *If* $(K, \sim) \equiv_{ij} (K', \sim')$, *and* $(K, \sim) \equiv_{ij} (K'', \sim'')$, *then* $(K', \sim') = (K'', \sim'')$.
4. *If* $(K, \sim) \in D_{ij}$, *then* $(K, \sim) \equiv_i (K', \sim') \iff \exists (K_1, \sim_1) \in \eta(\Gamma) : (K, \sim) \equiv_j (K_1, \sim_1) \wedge (K', \sim') \equiv_{ij} (K_1, \sim_1)$.
5. $s_{ij}(\eta(\Gamma)) = \eta(\Gamma)$.

Theorem

For any graph Γ, $\mathfrak{B}(\Gamma)$ *is a simple* QEA_n.

Theorem

For any graph Γ, $\mathfrak{B}(\Gamma)$ *is a simple* QEA_n.

Proof.

Omitted from the presentation, but it is very technical. $\square$

Definition

Let $\mathfrak{C}(\Gamma)$ be the subalgebra of $\mathfrak{B}(\Gamma)$ generated by the set of atoms.

Definition

Let $\mathfrak{C}(\Gamma)$ be the subalgebra of $\mathfrak{B}(\Gamma)$ generated by the set of atoms.

Note that the cylindric algebra constructed in [3] is $\mathfrak{Rd}_{ca}\mathfrak{B}(\Gamma)$ not $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$, but all results in [3] can be applied to $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$. Therefore, since our results depends basically on [3], we will refer to [3] directly when we apply it to catch any result about $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$.

Theorem

$\mathfrak{C}(\Gamma)$ *is a simple* QEA_n *generated by the set of the* $n-1$ *dimensional elements.*

Theorem

$\mathfrak{C}(\Gamma)$ is a simple QEA_n generated by the set of the $n-1$ dimensional elements.

Proof.

$\mathfrak{C}(\Gamma)$ is a simple QEA_n from Theorem 2.1. It remains to show that $\{(K,\sim)\} = \prod\{c_i\{(K,\sim)\} : i < n\}$ for any $(K,\sim) \in H$. Let $(K,\sim) \in H$, clearly $\{(K,\sim)\} \leq \prod\{c_i\{(K,\sim)\} : i < n\}$. For the other direction assume that $(K',\sim') \in H$ and $(K,\sim) \neq (K',\sim')$. We show that $(K',\sim') \notin \prod\{c_i\{(K,\sim)\} : i < n\}$. Assume toward a contradiction that $(K',\sim') \in \prod\{c_i\{(K,\sim)\} : i < n\}$, then $(K',\sim') \in c_i\{(K,\sim)\}$ for all $i < n$, i.e., $K'(i) = K(i)$ and $\sim' \restriction (n \setminus \{i\}) = \sim \restriction (n \setminus \{i\})$ for all $i < n$. Therefore, $(K,\sim) = (K',\sim')$ which makes a contradiction, and hence we get the other direction. $\square$

Theorem

Let Γ be a graph.

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*
2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Theorem

Let Γ be a graph.

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*
2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Proof.

1. We have $\mathfrak{Rd}_{ca}\mathfrak{C}$ is representable (c.f., [3]). Let $X = \{x \in \mathfrak{C} : \Delta x \neq n\}$. Call $J \subseteq \mathfrak{C}$ inductive if $X \subseteq J$ and J is closed under infinite unions and complementation. Then $\mathfrak{C}$ is the smallest inductive subset of $\mathfrak{C}$. Let f be an isomorphism of $\mathfrak{Rd}_{ca}\mathfrak{C}$ onto a cylindric set algebra with base U.

$\square$

Theorem

Let Γ be a graph.

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*
2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Proof.

1. Clearly, by definition, f preserves s_j^i for each $i, j < n$. It remains to show that f preserves s_{ij} for every $i, j < n$. Let $i, j < n$, since s_{ij} is boolean endomorphism and completely additive, it suffices to show that $fs_{ij}x = s_{ij}fx$ for all $x \in At\mathfrak{C}$. Let $x \in At\mathfrak{C}$ and $\mu \in n \setminus \Delta x$.

$\square$

Theorem

Let Γ be a graph.

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*
2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Proof.

1. If $\kappa = \mu$ or $l = \mu$, say $\kappa = \mu$, then

$$fs_{\kappa l}x = fs_{\kappa l}c_\kappa x = fs_l^\kappa x = s_l^\kappa fx = s_{\kappa l}fx.$$

If $\mu \notin \{\kappa, l\}$ then

$$fs_{\kappa l}x = fs_\mu^l s_l^\kappa s_\kappa^\mu c_\mu x = s_\mu^l s_l^\kappa s_\kappa^\mu c_\mu fx = s_{\kappa l}fx.$$

$\square$

Theorem

Let Γ be a graph.

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*
2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Proof.

2. Assume toward a contradiction that $\mathfrak{Rd}_{df}\mathfrak{C}$ is representable. Since $\mathfrak{Rd}_{ca}\mathfrak{C}$ is generated by $n-1$ dimensional elements then $\mathfrak{Rd}_{ca}\mathfrak{C}$ is representable. But this contradicts Proposition 5.4 in [3].

$\square$

Outline

Theorem

Let $2 < n < \omega$ and $\mathcal{T}$ be any signature between Df_n and QEA_n. Then the class of strongly representable atom structures of type $\mathcal{T}$ is not elementary.

Proof.

By Erdös's famous 1959 Theorem [4], for each finite κ there is a finite graph G_κ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let Γ_κ be the disjoint union of the G_l for $l > \kappa$. Clearly, $\chi(\Gamma_\kappa) = \infty$. So by Theorem 2.3 (1), $\mathfrak{C}(\Gamma_\kappa) = \mathfrak{C}(\Gamma_\kappa)^+$ is representable. $\square$

Proof.

Now let Γ be a non-principal ultraproduct $\prod_D \Gamma_\kappa$ for the Γ_κ. It is certainly infinite. For $\kappa < \omega$, let σ_κ be a first-order sentence of the signature of the graphs. stating that there are no cycles of length less than κ. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Łoś's Theorem, $\Gamma \models \sigma_\kappa$ for all κ. So Γ has no cycles, and hence by, [3] Lemma 3.2, $\chi(\Gamma) \leq 2$. By Theorem 2.3 (2), $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable. It is easy to show (e.g., because $\mathfrak{C}(\Gamma)$ is first-order interpretable in Γ, for any Γ) that

$$\prod_D \mathfrak{C}(\Gamma_\kappa) \cong \mathfrak{C}(\prod_D \Gamma_\kappa).$$

$\square$

Proof.

Combining this with the fact that: for any n-dimensional atom structure $\mathcal{S}$

$$\mathcal{S} \text{ is strongly representable} \iff \mathfrak{Cm}\mathcal{S} \text{ is representable},$$

the desired follows. $\square$

Outline

- Leon Henkin, J.Donald Monk, and Alfred Tarski, Cylindric algebras, part I, II, North-Holland, publishing company, Amesterdam London.
- I. Sain and R. Thompson. Strictly finite schema axiomatization of Quasi-Polyadic algebras. In "Algebraic Logic". Editors: H. Andreka, J. Monk and I. Nemeti. North Holland 1989.
- R. Hirsch and I. Hodkinson. Strongly representable atom structures of cylindric algebras. J. Symbolic Logic, 74:811-828, 2009.
- P. Erdös, Graph theory and probability, Canadian Journal of Mathematics, vol. 11 (1959), pp. 34-38.
- R. Diestel. Graph theory, volume 173 of Graduate Texts in Mathematics. Springer-Verlag, Berlin, 1997.