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Classification of absorbent-continuous, sharp FL_e-algebras on weakly real chains

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THE FIRST RESULT

- Every cancellative, Archimedean, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers.

  [O. Hölder, Die Axiome der Quantität und die Lehre vom Mass, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Classe, 53 (1901), 1– 64.]

Dropping isotonicity

- A continuous semigroup operations over intervals of real numbers is order isomorphic to a subsemigroup of the additive semigroup of the real numbers iff it is cancellative (reducible)

  [J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York-London, 1966.]

A epigammatic proof shows isotonicity

(AD) $\alpha * \beta \leq \gamma \iff \alpha \leq \beta \to \gamma$

(i) $\alpha * (\alpha \to \beta) \leq \beta$, $\beta \leq \alpha \to (\alpha * \beta)$

(ii) $(L, \leq, *)$ is a partially ordered monoid.

In the case of $\alpha \leq \beta$ we derive from Assertion (i) and the transitivity of $\leq$ the following inequality

$$\alpha \leq \gamma \to (\beta * \gamma) .$$

Applying (AD) we obtain $\alpha * \gamma \leq \beta * \gamma$; hence $(L, \leq, *)$ is a partially ordered monoid ([4]) . Because of (i) and (ii) the inequalities

- [G. Birkhoff, Lattice Theory, Amer. Math Soc. Colloquium Publications, 1973.]

Dropping Cancellation

- Every Archimedean, naturally and totally ordered semigroup in which the cancellation law does not hold can be embedded into either the real numbers in the interval $[0, 1]$ with the usual ordering and $ab = \max(a + b, 1)$ or the real numbers in the interval $[0, 1]$ and the symbol ∞ with the usual ordering and $ab=a+b$ if $a+b\leq 1$ and $ab=\infty$ if $a+b>1$.
- [A. H. Clifford, Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.]

Dropping cancellation

Commutativity came free

- Every Archimedean, naturally and totally ordered semigroup in which the cancellation law does not hold can be embedded into either th[illegible]al numbers in the interval [0, 1] with [illegible]ing and ab = max(a + b [illegible]bers in the interv[illegible]ymbol ∞ with the usual orderin[illegible] ab=a+b if a+b≤1 and ab=∞ if a+b>1.
- [A. H. Clifford, Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.]

Dropping Archimedean property

- Every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups

  [A. H. Clifford, Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.]

Dropping the total order

- A (*topological*) *semigroup* is a nonempty Hausdorff space with a (jointly) continuous and associative multiplication.
- compact, connected, with identity

The Theory of Compact Semigroups

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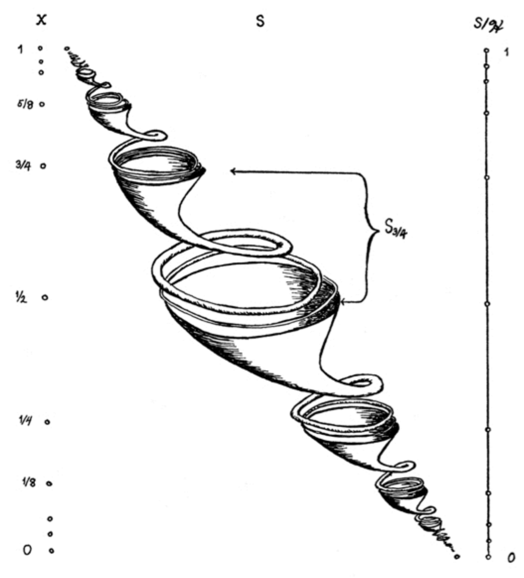
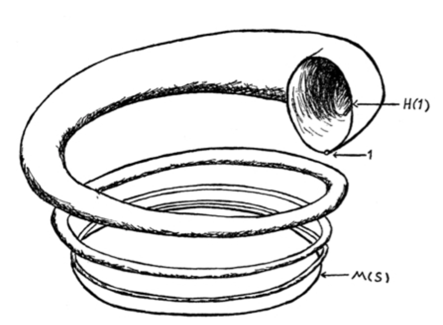
- compact, connected, with identity

The Theory of Compact Semigroups





The Theory of Compact Semigroups

- Topological semigroups over compact manifolds with connected, regular boundary B such that B is a subsemigroup: a subclass of compact connected Lie groups and via classifying (I)-semigroups, that is, semigroups on arcs such that one endpoint functions as an identity for the semigroup, and the other functions as a zero.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math., 65 (1957), 117–143.]

The Theory of Compact Semigroups

- (I)-semigroups are ordinal sums of three basic multiplications which an arc may possess.

  The word ‘topological’ refers to the continuity of the semigroup operation with respect to the topology.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math., 65 (1957), 117–143.]

(I)-SEMIGROUPS

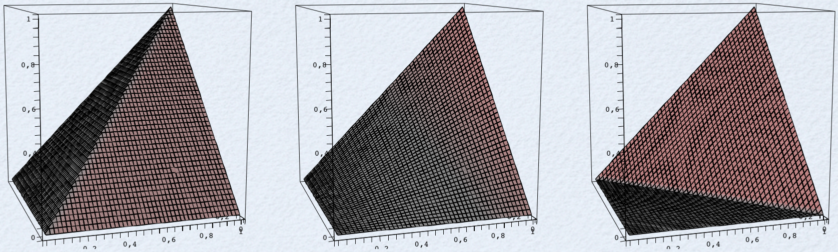


Figure 1: Minimum (left), product (center) and Łukasiewicz t-norms (right)

Naturally ordered case

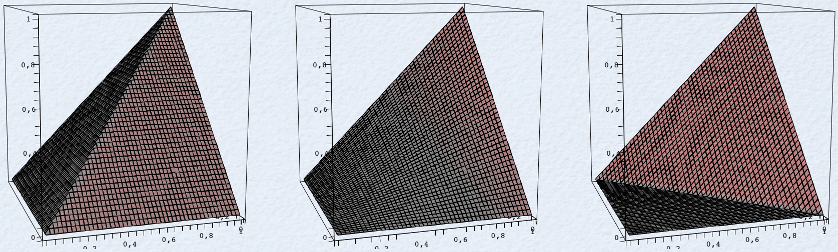


Figure 1: Minimum (left), product (center) and Łukasiewicz t-norms (right)

Divisible Case

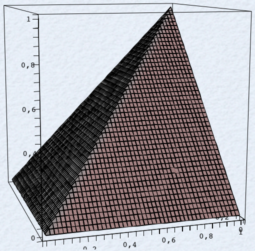


Figure 1: Minimum (left), product (center) and Łukasiewicz t-norms (right)

- Basic Logic

  [P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.]

Residuated posets

For any commutative binary operation $\circledast$ on a poset M the operation $\rightarrow_{\circledast} : M \times M \rightarrow M$ is called the residuum of $\circledast$ if for all $x, y, z \in M$ the following (adjointness) condition is satisfied: $x \circledast y \leq z \ \Leftrightarrow \ x \rightarrow_{\circledast} z \geq y$. Equivalently, $x \rightarrow_{\circledast} y$ is the largest $z \in M$ for which $x \circledast z \leq y$ holds. In this case M is called a commutative, residuated poset.

Residuated Lattices

$$\text{(AD)} \qquad \alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma$$

- Substructural Logics

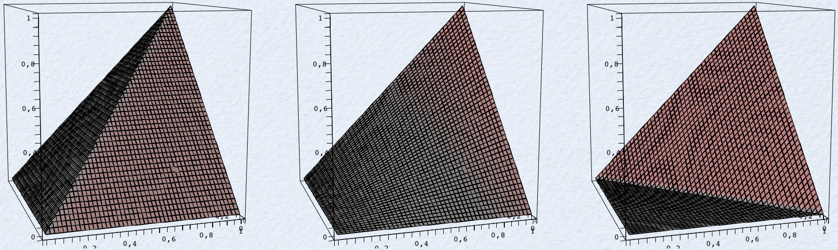
  [Galatos, N., Jipsen, P., Kowalski, T., & Ono, H. (2007). Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151. Studies in Logic and the Foundations of Mathematics, 532.]

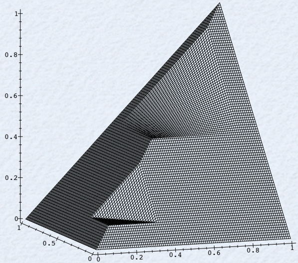
Residuated Lattices

- Substructural Logics:
  classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic, and their non-commutative versions

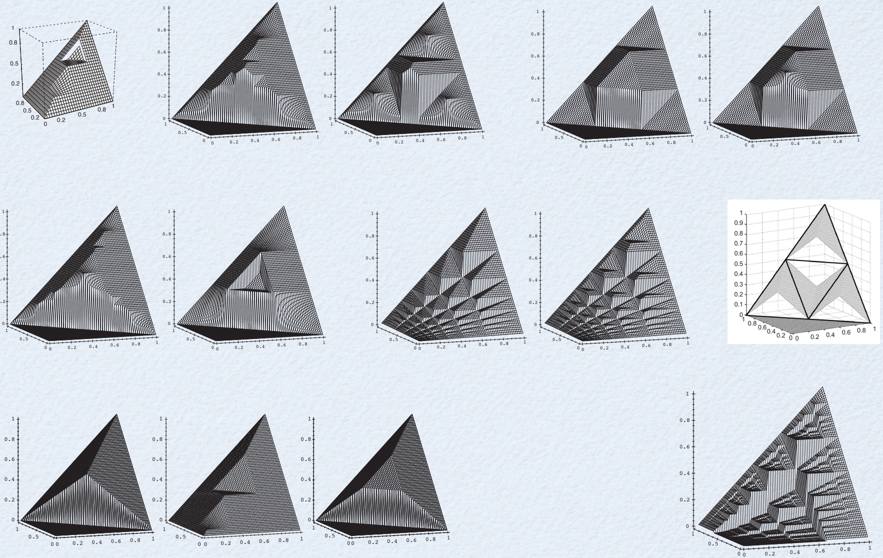
Example Integral Residuated Lattices on [0,1]

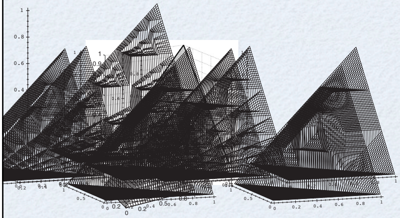
CONTINUOUS CASE

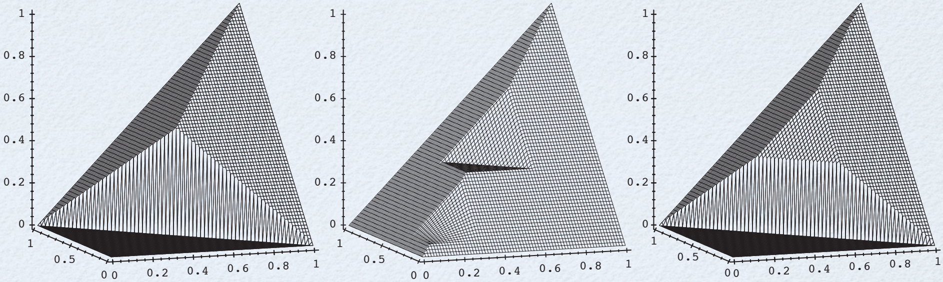




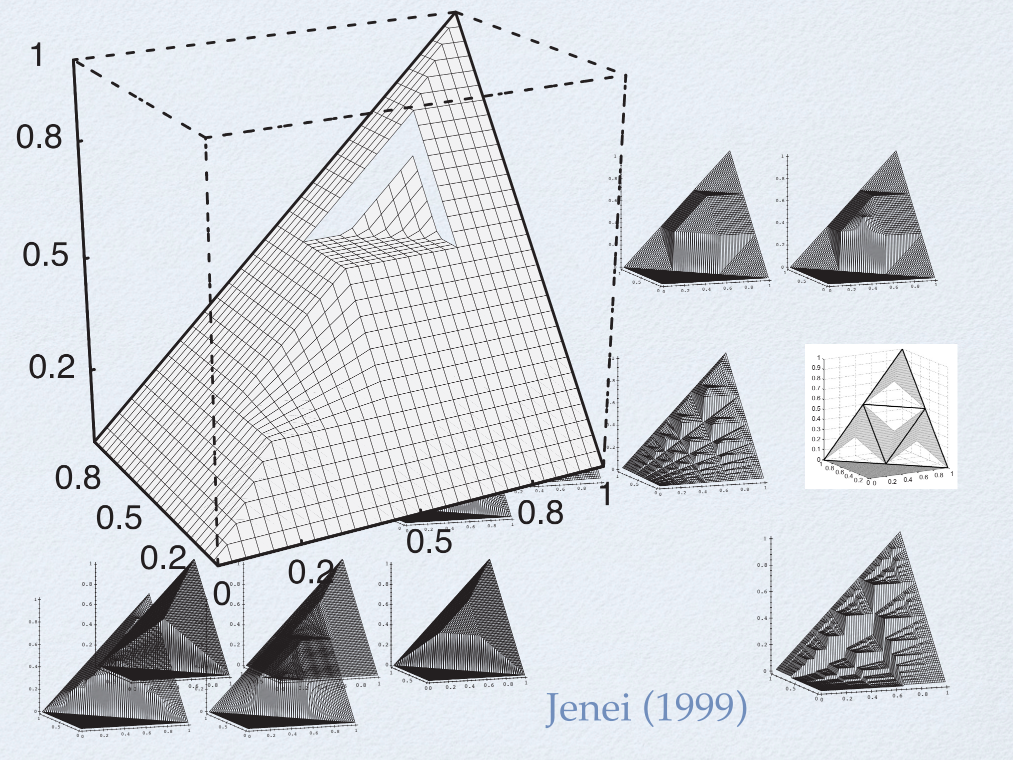
LEFT-CONTINUOUS CASE



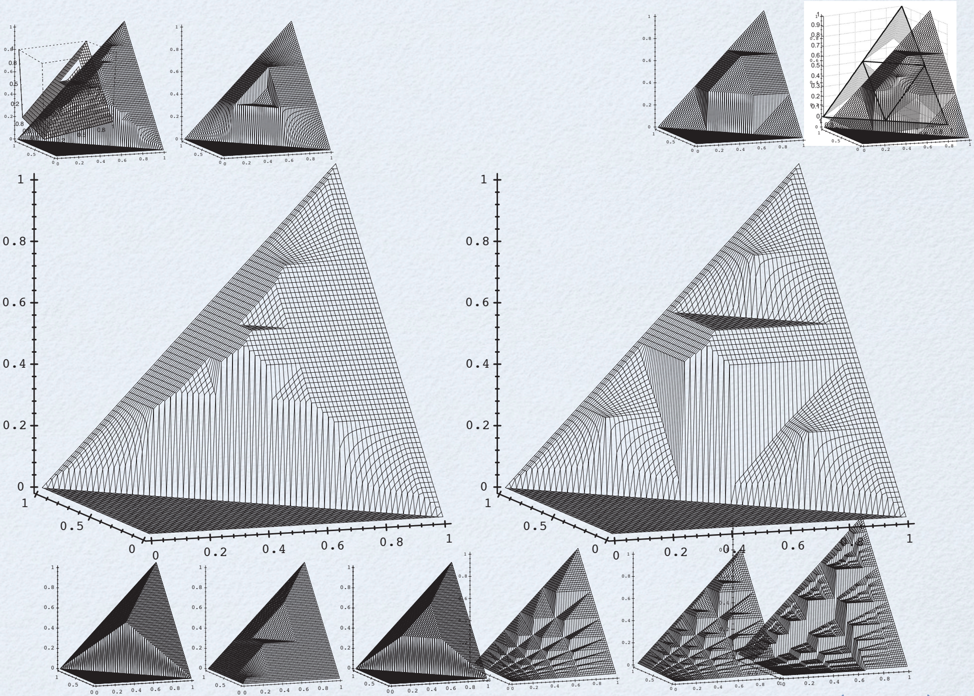




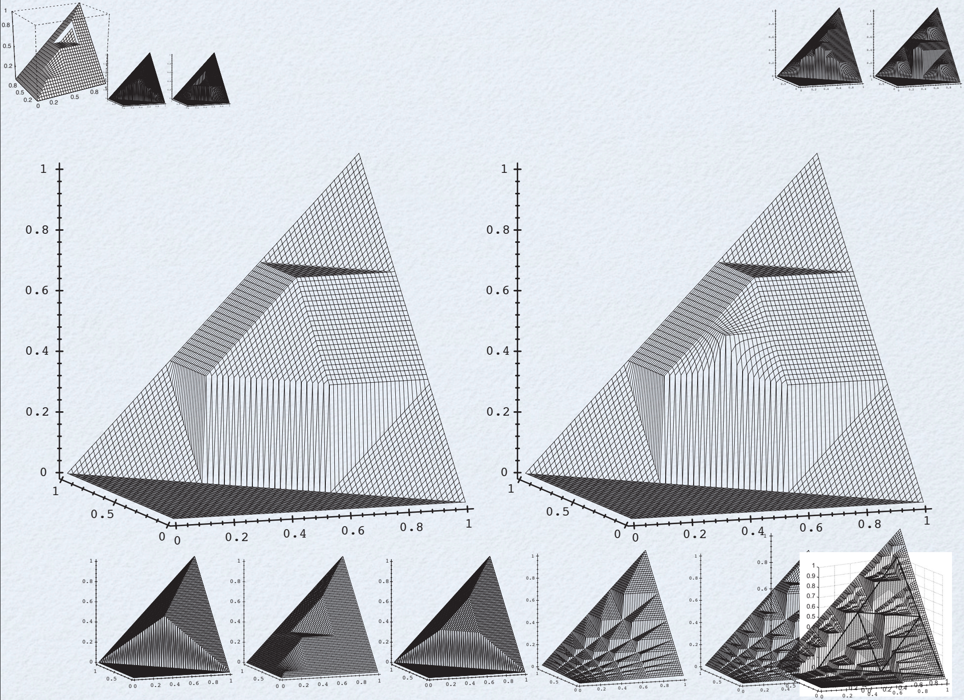
Jenei (2000)



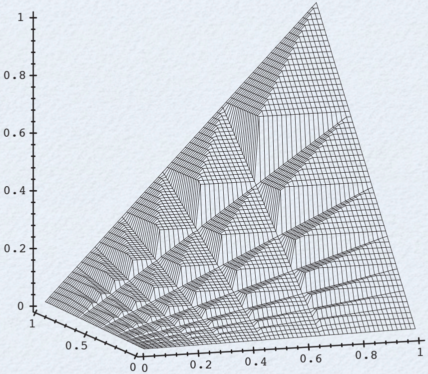
Jenei (1999)

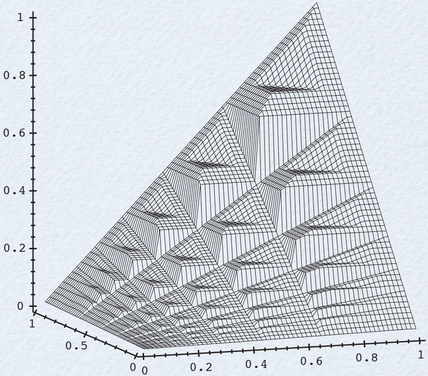


Jenei (2000)

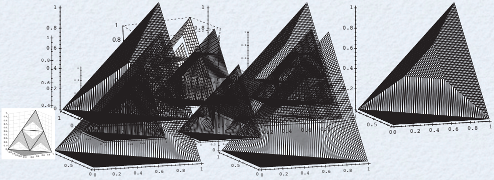


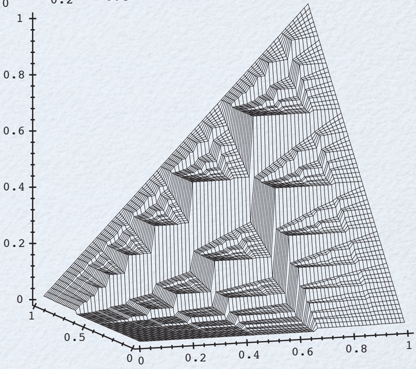
Jenei (2001)



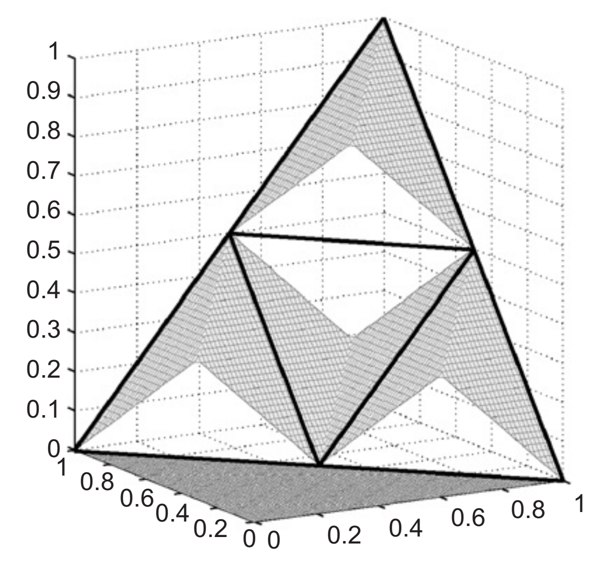


Jenei-Montagna (2003)

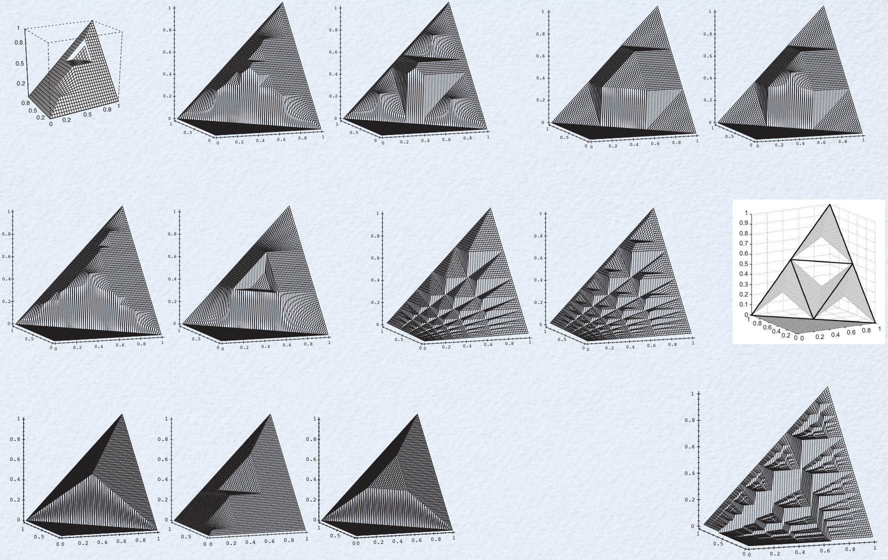




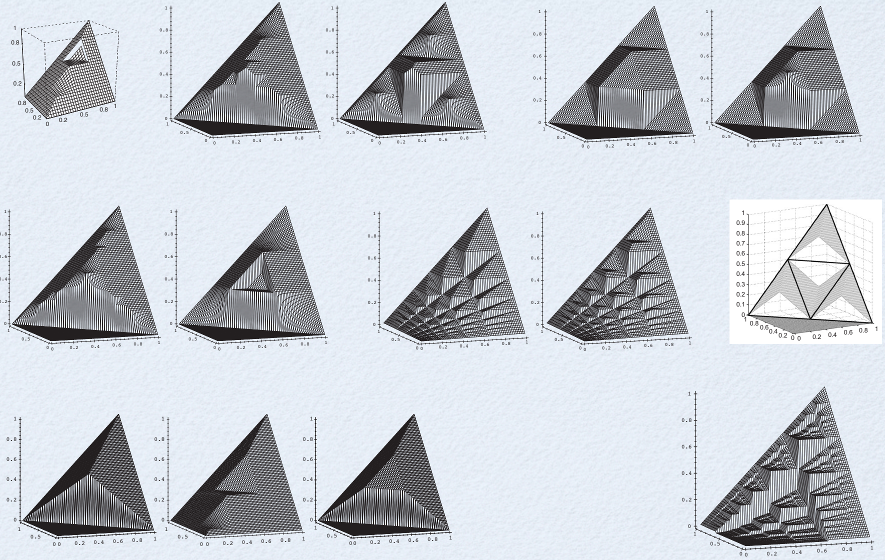
Maes-De Baets (2007)



Example Residuated Lattices



AN UNCHARTABLE WILDERNESS



Dropping topological assumptions

- BL-algebra = divisible + representable integral residuated lattice

- BL-algebras are subdirect poset products of MV-chains and product chains.

  [P Jipsen, F. Montagna, Embedding theorems for normal GBL-algebras, Journal of Pure and Applied Algebra, to appear]

Involutive Residuated Lattices

- Residuated lattice = lattice + residuated monoid (t its neutral element)

- FL_e-algebra = RL + f (f is an arbitrary constant)

- Involutive residuated lattice = RL + $x'' = x$
  where $x' = x \to f$

- Sharp RL = IRL + $f = t$

- SIU-algebra = bounded, representable, **sharp** RL +
  for $x, y \in X^-$, $x' \circledast y' = (x \circledast y)'$

Conic Representatioon

- Conic representation: For any conic, IRL

$$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^+ \\ x \otimes y & \text{if } x, y \in X^- \\ (x \to_\oplus y')' & \text{if } x \in X^+, y \in X^-, \text{ and } x \leq y' \\ (y \to_\otimes x')' & \text{if } x \in X^+, y \in X^-, \text{ and } x \not\leq y' \\ (y \to_\oplus x')' & \text{if } x \in X^-, y \in X^+, \text{ and } x \leq y' \\ (x \to_\otimes y')' & \text{if } x \in X^-, y \in X^+, \text{ and } x \not\leq y' \end{cases}$$

- [S. Jenei, H. Ono, On Involutive FL_e-monoids, Archive for Mathematical Logic, 2012, DOI: 10.1007/s00153-012-0295-6]

SIU-chains

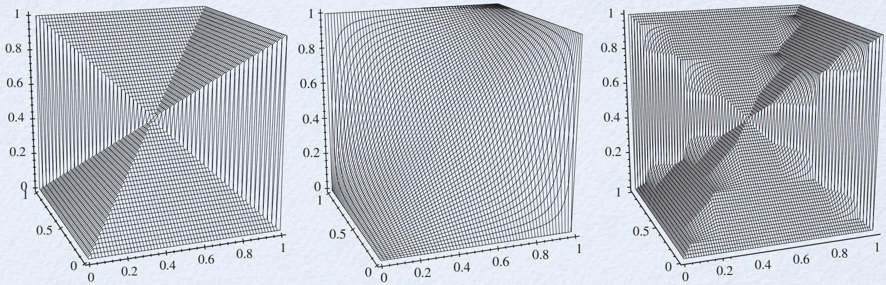
- 1) its negative cone is a BL-chain with only cancellative components and 2-element MV-components, and with no two consecutive cancellative ones,
  2) its positive cone is dual to its negative cone,
  3) the multiplication is given by

$$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^+ \\ x \otimes y & \text{if } x, y \in X^- \\ (x \to_\oplus y')' & \text{if } x \in X^+,\, y \in X^-, \text{ and } x \leq y' \\ (y \to_\otimes x')' & \text{if } x \in X^+,\, y \in X^-, \text{ and } x \not\leq y' \\ (y \to_\oplus x')' & \text{if } x \in X^-,\, y \in X^+, \text{ and } x \leq y' \\ (x \to_\otimes y')' & \text{if } x \in X^-,\, y \in X^+, \text{ and } x \not\leq y' \end{cases}$$

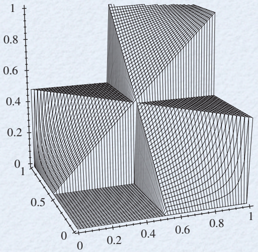
  [S. Jenei, On the relationship between the rotation construction and ordered Abelian groups, Fuzzy Sets and Systems 161 (2010), 277–284.]

  [S. Jenei, F. Montagna, Strongly Involutive Uninorm Algebras, Journal of Logic and Computation 2012; DOI: 10.1093/logcom/exs019]

ALL SIU-CHAINS



SIU-CHAIN - 2



1
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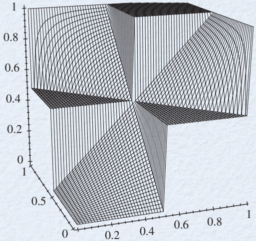

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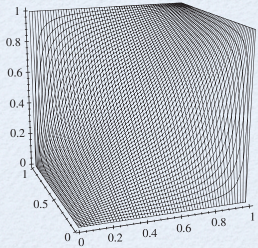

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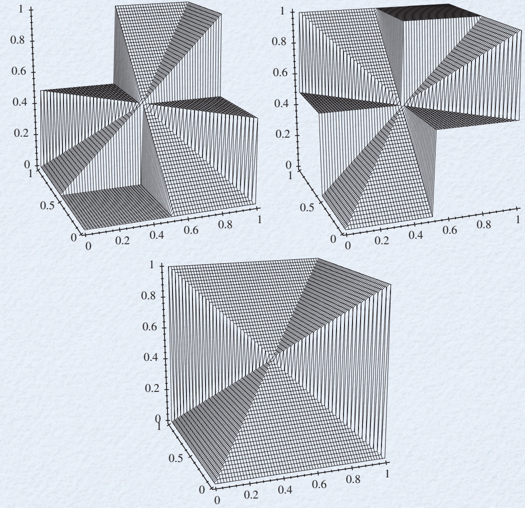
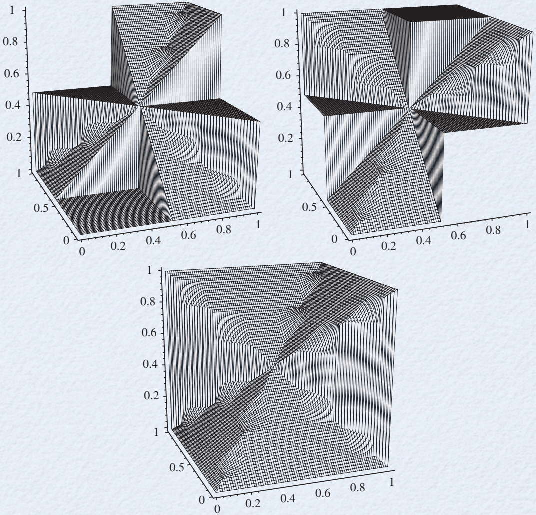
SIU-CHAIN - 1



SIV-CHAIN - 3



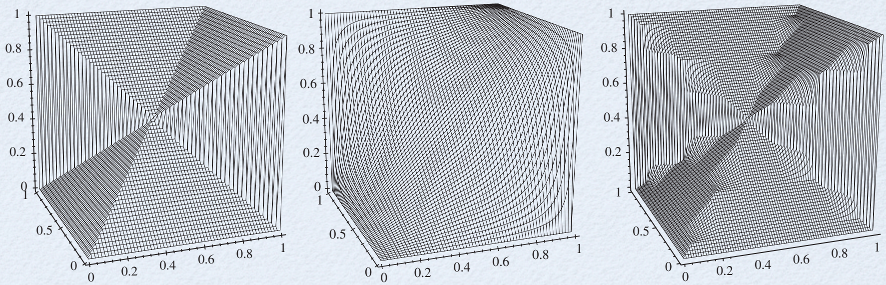
Absorbent continuous sharp residuated monoids on weakly real chains

- Call a chain $\langle X, \leq \rangle$ *weakly real* if X is order-dense and complete, there exists a dense $Y \subset X$ with $|Y| < |X|$, and for any $x, y \in Y$ there exist $u, v \in Y$ such that $u > x, v > y$, and there exists a strictly increasing function from $[x, u]$ into $[y, v]$.
- For $x \in X^{-}$, $a(x) = inf\{ u \in X^{-} : u \otimes x = x\}$
  Absorbent continuity = for $x \in X^{-}$, $a(x) \otimes x = x$

Absorbent continuous sharp residuated monoids on weakly real chains

- 1) its negative cone is a BL-algebra with only cancellative components and 2-element MV-components, and with no two consecutive cancellative ones,
  2) its positive cone is dual to its negative cone,
  3) the multiplication is given by
  $$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^+ \\ x \otimes y & \text{if } x, y \in X^- \\ (x \to_\oplus y')' & \text{if } x \in X^+,\ y \in X^-, \text{ and } x \leq y' \\ (y \to_\otimes x')' & \text{if } x \in X^+,\ y \in X^-, \text{ and } x \not\leq y' \\ (y \to_\oplus x')' & \text{if } x \in X^-,\ y \in X^+, \text{ and } x \leq y' \\ (x \to_\otimes y')' & \text{if } x \in X^-,\ y \in X^+, \text{ and } x \not\leq y' \end{cases}$$
  [S. Jenei, F. Montagna, Classification of absorbent-continuous, sharp FL_e-algebras on weakly real chains, submitted]

Absorbent continuous sharp residuated monoids on weakly real chains

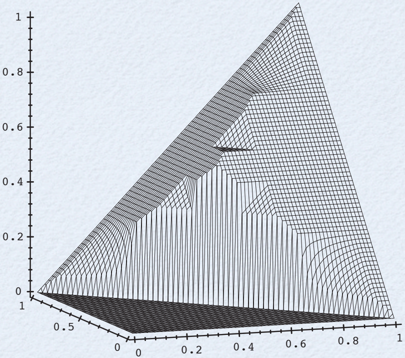


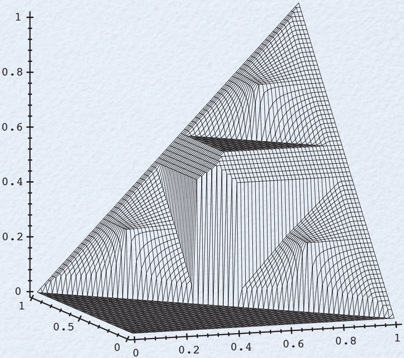
Absorbent Continuous Sharp Residuated Monoids on Weakly Real Chains

- Sharp residuated monoids on weakly real chains has this representation $\Leftrightarrow \otimes$ is absorbent continuous
- Absorbent continuity is a most relaxed version of the naturally ordered condition such that the algebra has this form.

Absorbent continuous sharp residuated monoids on weakly real chains

- Postulating the involutivity condition does not help at all in the integral case.





Thank you for your attention!