Decidability of The Tense Logic of Two Dimensional Minkowski Spacetime

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Abstract

We make a small step forward in the long running saga of understanding the temporal logics of Minkowski space-time by proving that the set of valid formulas over two-dimensional space-time is decidable.

Some interesting aspects include the use of Németi mosaics and the connection with interval temporal logics.

Warning

The main decidability result here has not been published. The full proof has not been subjected to independent peer review yet. Submission is not expected for a few months yet.

Apologies if mistakes are found later and we need to retract the claim.

Web site to announce updates on publication:
`www.csse.uwa.edu.au/~mark/research/Online/rel.htm`

A puli on the beach: so is it Hungary or Australia?

Outline

Introduction

Arthur Prior, pioneer of tense logic, realised that relativity challenged some of the basic assumptions that seem to underlie such logics. His last published talk before his untimely death in Norway (in 1969) was on this very subject [Pri70].

See [Mül07] for discussion of the interplay of tense logic and relativity at that time.

Prior would have wanted the tense logic of space-time to be well understood and well behaved. Over the decades there has not been much progress in this direction.

Note: we use terms *tense* and *temporal* logic interchangibly.

...

We consider Minkowski space-time, the set of all point-events of space-time under the relation of causal accessibility: u can access v if an electromagnetic signal could be sent from u to v.

We use Prior's tense language of F and P representing reflexive causal accessibility and its converse relation.

It is not known if this logic is decidable or even axiomatisable and this has been an open problem for decades.

Related earlier work by Rob Goldblatt showed that the dimension of the Minkowski frame can affect such properties of the tense logic.

Minkowski Spacetime

Fix a dimension $n \geq 2$. We will be looking at n-dimensional Minkowski Space Time and mostly concentrating on the case of $n = 2$ (despite $n = 4$ being more important for physics).

Time-points, or “events", in space time are just elements of $\mathbb{R}^n$.

The ordering on events is as follows. For $x = (x_1, ...x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$, we put $x \leq y$ iff $\Sigma_{i=1}^{n-1}(y_i - x_i)^2 \leq (y_n - x_n)^2$ and $x_n \leq y_n$.

We call $(\mathbb{R}^n, \leq)$ under this ordering $\mathbb{T}^n$.

Minkowski structures

Fix a countable set L of propositional atoms.

Propositions may or may not hold of events. Formalised via a standard temporal *valuation*, a map h from L to $\mathbb{R}^n$. The idea is that $p \in L$ holds at $x \in \mathbb{R}^n$ iff $x \in h(p)$.

An n-dimensional Minkowski Structure is $\mathcal{T} = (\mathbb{T}^n, \leq, h)$ for some valuation h.

The Logic: syntax

Formulas are built recursively from the atoms in L via classical $\neg$, $\wedge$ and Prior's temporal F and P.

But we use reflexive versions: $F\alpha$ means α occurs at some event accessible from now (including now).

Abbreviations $G\alpha = \neg F \neg \alpha$ (“always will be") and $H\alpha = \neg P \neg \alpha$ (“always has been") plus usual classical abbreviations.

Semantics

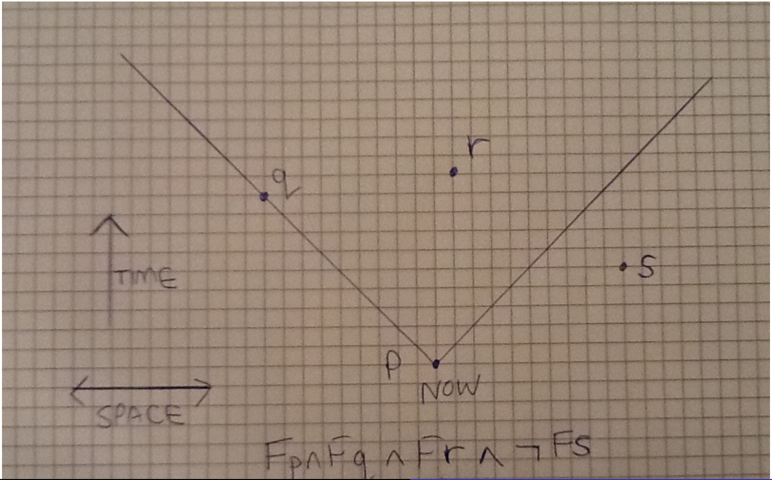
Formulas evaluated at points in structures $\mathcal{T} = (\mathbb{T}^n, \leq, h)$.

$\mathcal{T}, x \models \alpha$ means α is true at the point $x \in T$:

Semantic clauses:

$\mathcal{T}, x \models p$	iff	$x \in h(p)$, $(p \in L)$;
$\mathcal{T}, x \models \neg\alpha$	iff	$\mathcal{T}, x \not\models \alpha$;
$\mathcal{T}, x \models \alpha \wedge \beta$	iff	$\mathcal{T}, x \models \alpha$ and $\mathcal{T}, x \models \beta$;
$\mathcal{T}, x \models F\alpha$	iff	there is some $y \geq x$ such that $\mathcal{T}, y \models \alpha$;
$\mathcal{T}, x \models P\alpha$	iff	there is some $y \leq x$ such that $\mathcal{T}, y \models \alpha$;

Example Minkowski structure



The logic may be called $FP/\mathbb{T}^2$.

Say that a formula is *satisfiable* iff there is some structure $\mathcal{T}$ and some event x such that $\mathcal{T}, x \models \alpha$.

A formula is *valid* iff for all structures $\mathcal{T}$ and all events x we have $\mathcal{T}, x \models \alpha$.

We say that $FP/\mathbb{T}^n$ is *decidable* iff there is a finitely terminating algorithm which, on input any ϕ from the language, can answer whether ϕ is satisfiable or not.

Example Valid Formulas 1

From S4.2:

ax1) $G(p \to q) \to (Gp \to Gq)$

ax2) $Gp \to GGp$

ax3) $FGp \to GFp$

ax4) $Gp \to p$

Plus all the past versions.

Example Valid Formulas 2

Plus others:

ax5) $p \to GPp$

ax6) $FPp \to PFp$

ax7) $FPFp \to PFp$

(and past)

Density for a reflexive order? Not usually needed.

Plus Dedekind completeness? No idea.

Examples: Logics vary with dimension

$$
\begin{array}{rl}
\gamma = & GH(\neg(p \wedge q) \wedge \neg(p \wedge Fq) \wedge \neg(p \wedge Pq)) \\
\wedge & GH(\neg(q \wedge r) \wedge \neg(q \wedge Fr) \wedge \neg(q \wedge Pr)) \\
\wedge & GH(\neg(r \wedge p) \wedge \neg(r \wedge Fp) \wedge \neg(r \wedge Pp)) \\
\wedge & FPp \wedge FPq \wedge FPr \\
\wedge & FP(\neg p \wedge Fp \wedge \neg Fq \wedge \neg Fr) \\
\wedge & FP(\neg q \wedge Fq \wedge \neg Fp \wedge \neg Fr) \\
\wedge & FP(\neg r \wedge Fr \wedge \neg Fp \wedge \neg Fq) \\
\wedge & GH(Fp \wedge Fq \rightarrow Fr) \\
\wedge & GH(Fp \wedge Fr \rightarrow Fq) \\
\wedge & GH(Fq \wedge Fr \rightarrow Fp)
\end{array}
$$

$\neg\gamma$ is valid for $n = 2$, not valid for $n > 2$.
Note: Goldblatt gave similar formulas for the irreflexive tense.

Reasoning

Axiomatisations: unknown.

Decidability: unknown.

Known results for other related logics:

Goldblatt showed that the “Diodorean" modal logic of these structures is S4.2 for all $n \geq 2$.

Shehtman and Shapirovsky investigated the case for a slower than light accessibility relation.

Phillips looked at modal and tense logics for $\mathbb{N}$ rather than $\mathbb{R}$.

Overview of Proof

We will show decidability for $FP/\mathbb{T}^2$.

To do so we introduce a type of finitely described object called a “mosaic" following the idea of Istvan Németi in [Ném95].

For a particular formula of interest there are only a finite number of mosaics to consider.

We show that satisfiability of the formula in a Minkowski structure is equivalent to finding a subset of those mosaics satisfying some checkable properties. Hence decidable.

Closure Set

Fix ϕ. We want to know if ϕ is satisfiable or not.

Define the closure set to contain all the subformulas ($\psi \leq \phi$) of ϕ and their negations: **clos**$(\phi) = \{\psi, \neg\psi | \psi \leq \phi\}$.

Identify $\neg\neg\alpha$ with α and we may assume that **clos**(ϕ) is closed under taking negations.

Colours

Definition

Suppose $\Gamma \subseteq \mathbf{clos}(\phi)$. Say Γ is a *(ϕ-)colour* if the following hold:

1. for all $\alpha \wedge \beta \in \mathbf{clos}(\phi)$, $\alpha \wedge \beta \in \Gamma$ iff both $\alpha \in \Gamma$ and $\beta \in \Gamma$; and
2. for all $\alpha \in \mathbf{clos}(\phi)$, $\neg\alpha \in \Gamma$ iff $\alpha \notin \Gamma$.

These are thus maximally propositionally consistent subsets of the closure set. Let C_ϕ be the set of all ϕ-colours.
We order C_ϕ as follows: $\Gamma \leq \Delta$ iff 1) for all $\neg F\alpha \in \mathbf{clos}(\phi)$, if $\neg F\alpha \in \Gamma$ then $\neg\alpha \in \Delta$; and 2) for all $\neg P\alpha \in \mathbf{clos}(\phi)$, if $\neg P\alpha \in \Delta$ then $\neg\alpha \in \Gamma$.

Clusters

Say that $\Gamma \equiv \Delta$ iff $\Gamma \leq \Delta$ and $\Delta \leq \Gamma$.

Also put $\Gamma \# \Delta$ iff neither $\Gamma \leq \Delta$ nor $\Delta \leq \Gamma$.

Let $\mathbf{Clus}_\phi = C_\phi / \equiv$ be the set of $\equiv$-classes, called *clusters*.

Order $\mathbf{Clus}_\phi$ via $c \leq d$ iff there exists $\Gamma \in c$ and $\Delta \in d$ such that $\Gamma \leq \Delta$.

Stories

For any $n \geq 1$, say that a sequence $v = \langle v_1, v_2, ..., v_{2n-1} \rangle$ of clusters and/or colours is a *(ϕ-)story* iff

1. for each i, $v_{2i-1} \in \mathbf{Clus}_\phi$ and $v_{2i} \in C_\phi$;
2. each $v_i \leq v_{i+1}$; and
3. for each i, $v_{2i-1} \neq v_{2i+1}$.

The *length* $|v|$ of the story $v = \langle v_1, v_2, ..., v_{2n-1} \rangle$ is n.

Let D_ϕ be the set of all stories. There are only finitely many.

Alignments

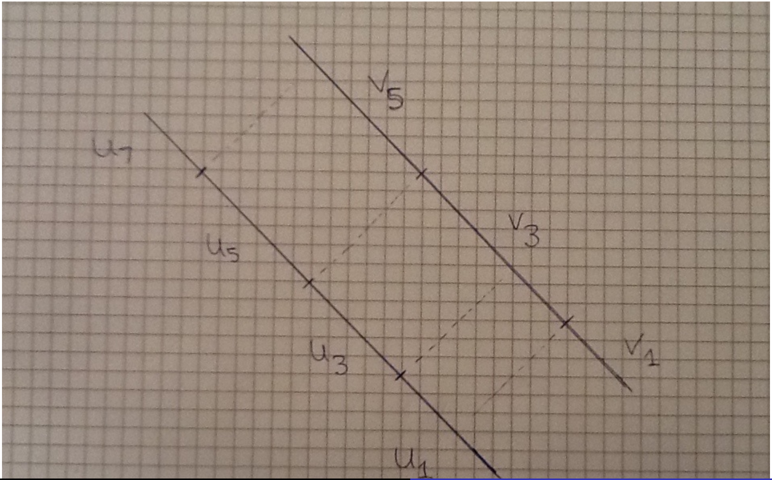
Given stories u and v we say that
$A \subseteq \{1, 2, ..., 2|u| - 1\} \times \{1, 2, ..., 2|v| - 1\}$ is an *alignment* of
$u \leq v$ iff :

1. for all i, there is some j such that $(i, j) \in A$;
2. for all j, there is some i such that $(i, j) \in A$;
3. if $(2i, j) \in A$ and $(2i, j') \in A$ then $j = j'$;
4. if $(i, 2j) \in A$ and $(i', 2j) \in A$ then $i = i'$;
5. if $(i, j) \in A$, $i < i'$ and $j' < j$ then we do not have $(i', j') \in A$; and
6. for all $(i, j) \in A$, $u_i \leq v_j$.

Mosaics

A *(ϕ-)mosaic* is a triple (u, A, v) such that A is an alignment of $u \leq v$.

a mosaic



Composition

Say that $m' = (u', A', v')$ and $m'' = (u'', A'', v'')$ compose to $m = (u, A, v)$ iff $u = u'$, $v = v''$, $v' = u''$ and there is some subset
$B \subseteq \{1, 2, ..., 2|u| - 1\} \times \{1, 2, ..., 2|v'| - 1\} \times \{1, 2, ..., 2|v| - 1\}$
such that:

1. for all i, j, $(i, j) \in A$ iff there is some k such that $(i, k, j) \in B$;
2. for all i, j, $(i, j) \in A'$ iff there is some k such that $(i, j, k) \in B$; and
3. for all i, j, $(i, j) \in A''$ iff there is some k such that $(k, i, j) \in B$.

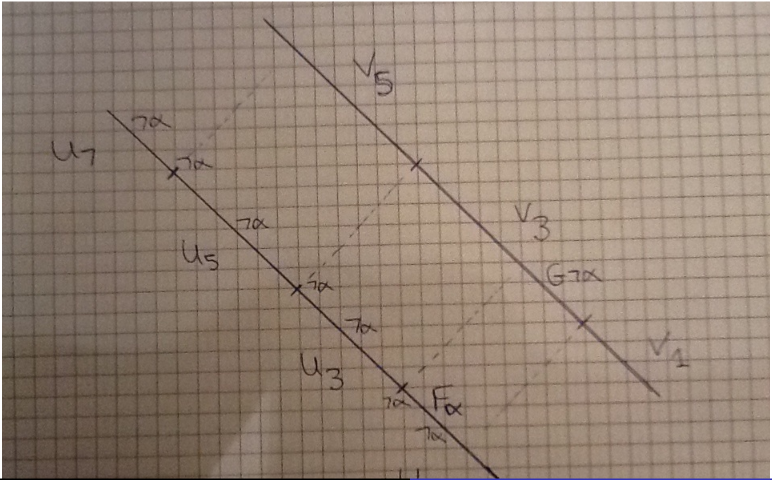
In that case, also say that m' and m'' *compose*.
Can generalise to any finite, non-empty sequence of mosaics.

Defects

A *defect* in a mosaic is either:

(F external)	$F\alpha$ appears in v_i and no α in any v_j for $j \geq i$;
(F internal)	$F\alpha$ appears in u_i, and there is no $k \geq i$ with α in u_k but there is some j with $(i, j) \in A$ and $\neg F\alpha$ in u_j;
(F eternity)	generic in every mosaic:
(P ...)	mirrors; and
(Density)	generic in every mosaic.

An F internal defect

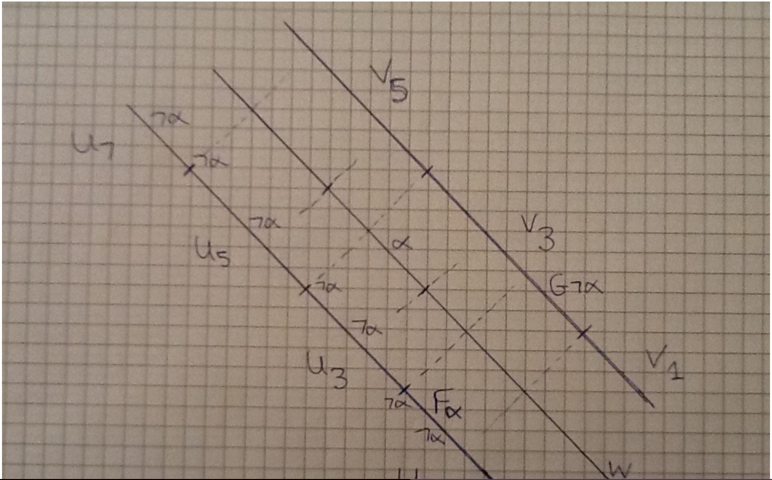


Cures

Cure for type F external defect: Suppose $m = (u, A, v)$ is a ϕ-mosaic and there is i such that $F\alpha$ in v_i and no α in any v_j for $j \geq i$. Thus $F\alpha$ is a F external defect in m. Then we say that the mosaic $m' = (u', A', v')$ is a *cure* for that defect iff $v = u'$ and there is $j \leq k$ such that α in u'_k and $(i, j) \in A'$.

Cure for type F internal defect: Suppose $m = (u, A, v)$ is a ϕ-mosaic and there is i such that $F\alpha$ appears in u_i, no $k \geq i$ with α in u_k but some j with $(i, j) \in A$, and $\neg F\alpha$ in v_j. Thus $F\alpha$ is a F-internal defect in m. Then we say that the mosaics $m' = (u', A', v')$ and $m'' = (u'', A'', v'')$ are a *cure* for that defect iff m' and m'' compose to m and there is k such that α in v'_k and $(i, k) \in A'$.

An F internal cure



More Cures

Cure for future eternity defect: Every ϕ-mosaic m has a future eternity defect. We say that the mosaic m' is a *cure* for that defect iff m and m' compose.
Cure for density defect: Every ϕ-mosaic m has a density defect. We say that the mosaics $m' = (u', A', v')$ and $m'' = (u'', A'', v'')$ are a *cure* for that defect iff m' and m'' compose to m.
Other cures similar.

SSMs

Definition

A *Saturated Set of Mosaics (SSM) M* for ϕ is a set of ϕ-mosaics such that:

- there is a mosaic (u, A, v) in M with ϕ in some Γ in some u_i;
- all defects in mosaics in M have cures by mosaics in M.

Finitely Separated

Definition

For each m in M, say $m = (u, A, v)$ is *finitely separated* (in M) iff there is a sequence $\langle m_1, m_2, ..., m_e \rangle$ of mosaics from M which compose to m such that for each $d < e$ there is some pair $(2i-1, 2j-1) \in A$ such that the mosaic m_d aligns some cluster $> u_{2i-1}$ with some cluster $< v_{2j-1}$.

Split

Definition

For each m in M, say m is *split* (in M) iff there are mosaics m_1, m_2, m_3, m_4 in M which compose to m and for all $i < |u|$, for all $j < |v|$, at least one of the following hold. 1) $i = |u|$. 2)$j = 1$.
3) if $(2i-1, 2j-1) \in A$ then there is some cluster $\leq u_{2i-1}$ along the shared edge of m_1/m_2 aligned by m_2 with some cluster at the shared edge of m_2/m_3 aligned by m_3 with some cluster $\geq v_{2j-1}$ along the shared edge of m_3/m_4.
4) if $(2i-1, 2j-1) \in A$ then there is some cluster $\leq u_{2i-1}$ along the shared edge of m_1/m_2 aligned by m_2 with some cluster at the shared edge of m_2/m_3, being the predecessor of some cluster at the shared edge of m_2/m_3, aligned by m_3 with some cluster $\geq v_{2j-1}$ along the shared edge of m_3/m_4.

DCSSM

Definition

A SSM M of ϕ-mosaics satisfies the *SSM Dedekind Completeness property* iff the following property holds. For each $m \in M$, either m is finitely separated or m is split. Call such a set a *DCSSM*.

Soundness

Lemma

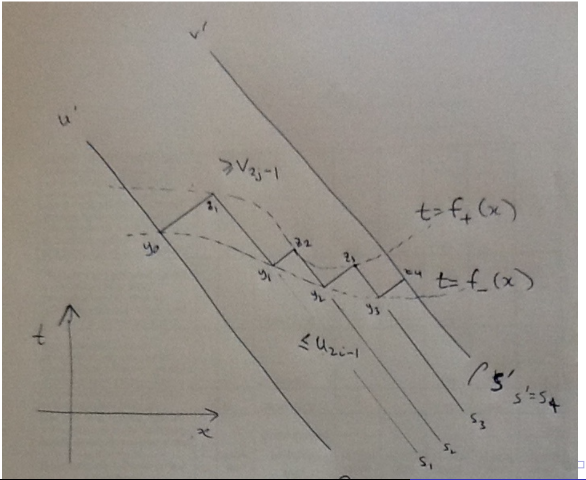
If ϕ is satisfiable then there is a DCSSM for ϕ.

Proof strategy:

If ϕ has a model then we can find actual mosaics in it for every pair of diagonals following leftward-heading photons.

Can show the set of such actual mosaics is a DCSSM.

Zig zags while separating a mosaic



Completeness

Lemma

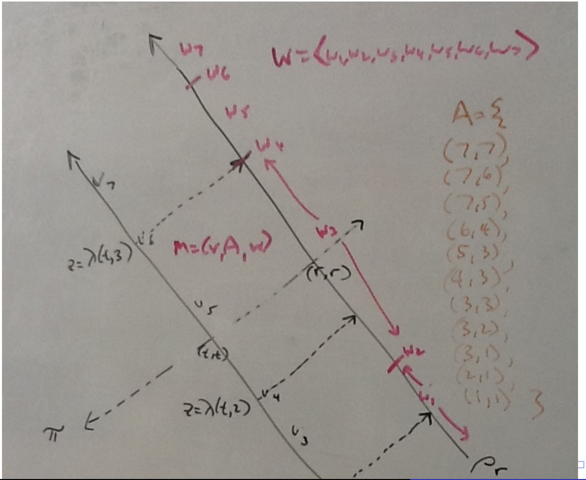
If there is an DCSSM for ϕ then ϕ is satisfiable.

Proof strategy:

Build a partially filled labelled chronicle (quasimodel, Henkin structure) for ϕ step by step by placing mosaics together, or inside each other, to cure defects. Colour in whole rational diagonals just densely within clusters.

DC property can be used to show that we can then easily pick suitable colours for all points along the irrational diagonals.

Curing an external defect



Decidability

Theorem

Satisfiability for $FP/\mathbb{T}^2$ is decidable.

Proof strategy: Start with the set of all the mosaics for ϕ and repeatedly throw away ones that do not have cures for defects in the current set.

Stop when there is no witness for ϕ or when the process stabilises.

Note: looks like at least double exponential time complexity.

Summary

We have used Németi style mosaics to show that the reflexive temporal logic of two-dimensional Minkowski space-time is decidable.

Future Work

Undecidable via tiling for $n \geq 3$.

Axiomatizations?

Until and since?

Irreflexive relations.

Related interval logic

Thank you

Thank you for listening.

Questions? Comments.

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