On generic automorphisms

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Németifest

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Motivation

G is a permutation group on a set of cardinality $\aleph_0$. What should be regarded as a typical member of G?

- G is Polish group ($\rightarrow$ Baire Category Theorem!)
- Basic open sets of the form $[p] = \{g \in G : g \text{ extends } p\}$, for p partial map.
- Typical elements should have any property which it would be unreasonable to prohibit on the basis of finitely many of its values.
- Typical elements should lie in certain dense open sets.

Motivation

Any two generic elements should look alike: they should be conjugates.

Definition (Truss) $g \in G$ is generic if it lies in a comeagre conjugacy class, that is, the complement of the conjugacy class

$$g^G = \{h^{-1}gh : h \in G\}$$

is the union of countably many nowhere dense sets.

Equivalently, g^G is dense G_δ.

Examples

Groups having a dense conjugacy class:

- Automorphism group of a standard Borel space (Rokhlin),
- Unitary group of a separable infinite-dimensional Hilbert space (Choksi, Nadkarni),
- Isometry group of the Urysohn space,
- Homeomorphism group of the Hilbert cube, or the Cantor space

Examples

Groups having a generic element

- $Aut(\mathbb{N}, =)$ (the symmetric group on a countable set).
- $Aut(\mathbb{Q}, <)$.
- $Aut(R)$, where R is the countable random graph.
- $Aut(P)$, where P is the countable random poset.

Usual context

- $\mathcal{M} = \langle M, \text{some relations} \rangle$ is a homogeneous, countable structure endowed with discrete topology.
- $G = Aut(\mathcal{M})$ automorphism group of $\mathcal{M}$.

$G \subseteq {}^{M}M$ is a Polish group with the topology inherited from the product topology of ${}^{M}M$.

g^G is dense if for any finite partial p there is $h \in G$ such that $h^{-1}gh \supseteq p$.

Theorem (Truss)

$\mathcal{M}$ has generic automorphism if and only if for all finite elementary mappings p and q there is an automorphism h such that $h^{-1}ph$ and q are compatible (i.e. $h^{-1}ph \cup q$ is elementary).

Proof idea

Define the poset $(P, \leq)$:

- $P =$ finite elementary mappings,
- $p \leq q$ if $q \subseteq h^{-1}ph$ for some automorphism h.

Dense sets:

- $D_x = \{p : \ x \in \mathrm{dom}(p)\}$, $R_y = \{p : \ y \in \mathrm{ran}(p)\}$,
- $E_p = \{q : \ h^{-1}ph \subseteq q$ for some automorphism $h\}$.

There exists a generic filter F (in the forcing sense). $\bigcup F$ is an generic automorphism.

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Remark: Applying certain forcing axioms the proof can be extended to automorphism groups of not necessarily countable structures.

Ample generics

Definition: G has ample generics if for each finite n there is a comeager orbit for the diagonal conjugacy action of G on G^n

$$g \cdot (g_1, \ldots, g_n) = (g^{-1} g_1 g, \ldots, g^{-1} g_n g).$$

Examples:

- $Aut(\mathbb{Q}, <)$ has a generic automorphism but does not have ample generics,
- Automorphism group of the countable random graph (Hrushovski),
- Free group on countably many generators (Bryant, Evans),
- Arithmetically saturated models of true arithmetic (Schmerl),
- Automorphism group of ω-stable, $\aleph_0$-categorical (Hodges, Hodkinson, Lascar, Shelah)

Polish groups

The existence of ample generics of G implies:

- The small index property,
(open subgroups are the ones with small index)
- Automatic continuity of homomorphisms,
(homomorphism into separable groups are automatically continuous)
- G admits a unique Polish group topology.

Chromatic number of the direct product

The chromatic number $\chi(D)$ of a graph D is the least cardinal κ for which there is a coloring with κ colors so that no adjacent vertices have the same color.

$H(\kappa, \lambda)$ is the statement: if $\chi(D), \chi(D') \geq \kappa$ then $\chi(D \times D') \geq \lambda$.

Easy: $\chi(D \times D') \leq \chi(D), \chi(D')$.

Chromatic number of the direct product

Hajnal: $H(\aleph_0, \aleph_0)$

Conjecture: $\exists n < \omega \; H(n, 4)$ (weak Hedetniemi conjecture).

Schmerl: If $\mathcal{M} \not\models TA$ then TFAE:

1. $\mathcal{M}$ has an automorphism whose set of conjugates is dense;
2. There is $n < \omega$ such that $\mathcal{M} \models H(n, 4)$.

Schmerl: If $\mathcal{M} \not\models TA$ is arithmetically saturated then TFAE:

1. $\mathcal{M}$ has a generic automorphism;
2. There is $n < \omega$ such that $\mathcal{M} \models H(n, 4)$.

Hrushovski-type theorems

Hrushovksi: Each finite graph G can be enlarged to another finite graph H such that every partial isomorphism of G extends to an automorphism of H.

Same for relational structures:

Herwig: Each finite relational structure G can be enlarged to a finite H such that every partial isomorphism of G extends to an automorphism of H.

What if we want not just arbitrary relation structures extending G?

Hrushovski-type theorems

Let **K** be an amalgamation class and $\mathcal{M}$ be the Fraïssé limit of **K**. Then the following are equivalent

1. Each $\mathcal{A} \in$ **K** can be embedded into some $\mathcal{B} \in$ **K** such that partial isomorphisms of $\mathcal{A}$ extends to automorphisms of $\mathcal{B}$.
2. $\mathcal{M}$ has a generic automorphism such that each finite n-tuple has a finite orbit (under the diagonal action).

Remark: This theorem can be reformulated as a pure semigroup theoretic statement.

Example: Countable random graph.

Lyndon's interpolation

Main message: A local version of Lyndon's interpolation theorem implies the existence of generic automorphisms.

Lyndon's interpolation: For first order theories S and T, if $S \cup T$ is unsatisfiable, then there is an interpolating sentence φ in the language of $S \cap T$ that is true in all models of S and false in all models of T.

Local version?

Local interpolation

An n-type of $\mathcal{M}$ over a subset $A \subseteq M$ is a finitely satisfiable set $p(\bar{x})$ of formulae with free variables $\bar{x}$ and parameters from A.

Typical example is the type of an n-tuple:

$$tp(\bar{a}/A) = \{\varphi(\bar{x}, \bar{c}) : \ \mathcal{M} \models \varphi[\bar{a}, \bar{c}], \ \bar{c} \in A\}$$

which describes the connection between $\bar{a}$ and elements of A.

A type p is algebraic if p can be realized by a finite number of elements only.

Fraïssé limits of amalgamation classes that are closed under disjoint unions (e.g. finite graphs) do not have algebraic types.

Local interpolation

Definition: $\mathcal{M}$ has the local interpolation property if for any two types p and q of $\mathcal{M}$ over a finite set, $p \cup q$ is satisfiable provided $p \cap q$ is.

Equivalently, if $p \cup q$ is not satisfiable then there is φ so that $\varphi \in p$ and $\neg\varphi \in q$.

Examples:

1. Random graph,
2. Random poset.

Local interpolation and generic automorphisms

Theorem (Gy–S): If $\mathcal{M}$ has no algebraic types and has the local interpolation property then it has a generic automorphism.

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Remark: No mention of partial mappings.

Thank you!