A new representation theory

Representing cylindric-like algebras by cylindric relativized set algebras

Miklós Ferenczi

TUB Department of Algebra

September 2012

CONTENTS

- On the Resek-Thompson-Andréka theorem (and on the refinement)
- On the results obtained after the theorem
- Connections with neat embeddability

Boolean algebras are representable, but
Cylindric algebras, Polyadic equality algebras not!

Definition

$\mathfrak{A} \in \mathrm{CA}_\alpha$ is representable if $\mathfrak{A} \in \mathbf{I}\,\mathrm{Gs}_\alpha$.

(the unit of a Gs_α is of the form $\cup_{k\in K}\, {}^{\alpha}U_k$ where $U_i \cap U_j = \emptyset$, $i \neq j$)

Theorem

(Monk) $I\,Gs_\alpha$ is not finite schema axiomatizable.

Important representable subclasses: Lf_α, Dc_α, Ss_α, etc.

Theorem

(Resek) $\mathfrak{A} \in CA_\alpha^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_\alpha \cap CA_\alpha)$.

where

- CA_α^{++} : the system of axioms of CA_α is expanded by infinitely many merry-go-round axioms
- for the unit V of a Crs_α, $V \subseteq {}^\alpha U$ for some set U
- Crs: Henkin, Németi, vanBenthem.

Theorem

(Resek) $\mathfrak{A} \in CA_\alpha^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_\alpha \cap\ CA_\alpha)$.

where

- CA_α^{++} : the system of axioms of CA_α is expanded by infinitely many merry-go-round axioms
- for the unit V of a Crs_α, $V \subseteq {}^\alpha U$ for some set U
- Crs: Henkin, Németi, vanBenthem.
- Concept of r-representation.

Theorem

(Resek) $\mathfrak{A} \in CA_\alpha^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_\alpha \cap CA_\alpha)$.

where

- CA_α^{++} : the system of axioms of CA_α is expanded by infinitely many merry-go-round axioms
- for the unit V of a Crs_α, $V \subseteq {}^\alpha U$ for some set U
- Crs: Henkin, Németi, vanBenthem.
- Concept of r-representation.
- Analogy in Logic: the completeness of the Henkin's semantics.

Theorem

(Resek) $\mathfrak{A} \in CA_\alpha^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_\alpha \cap CA_\alpha)$.

where

- CA_α^{++} : the system of axioms of CA_α is expanded by infinitely many merry-go-round axioms
- for the unit V of a Crs_α, $V \subseteq {}^\alpha U$ for some set U
- Crs: Henkin, Németi, vanBenthem.
- Concept of r-representation.
- Analogy in Logic: the completeness of the Henkin's semantics.
- PROBLEMS:

Theorem

(Resek) $\mathfrak{A} \in CA_{\alpha}^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_{\alpha} \cap CA_{\alpha})$.

where

- $\mathrm{CA}_{\alpha}^{++}$: the system of axioms of CA_{α} is expanded by infinitely many merry-go-round axioms

- for the unit V of a Crs_{α}, $V \subseteq {}^{\alpha}U$ for some set U

- Crs: Henkin, Németi, vanBenthem.
- Concept of r-representation.

- Analogy in Logic: the completeness of the Henkin's semantics.
- PROBLEMS:
- the infinite schema,

Theorem

(Resek) $\mathfrak{A} \in CA_\alpha^{++}$ *if and only if* $\mathfrak{A} \in I\,(Crs_\alpha \cap CA_\alpha)$.

where

- CA_α^{++} : the system of axioms of CA_α is expanded by infinitely many merry-go-round axioms
- for the unit V of a Crs_α, $V \subseteq {}^\alpha U$ for some set U
- Crs: Henkin, Németi, vanBenthem.
- Concept of r-representation.
- Analogy in Logic: the completeness of the Henkin's semantics.
- PROBLEMS:
- the infinite schema,
- the class $\mathrm{Crs}_\alpha \cap \mathrm{CA}_\alpha$.

Theorem

(Resek-Thompson-Andréka) $\mathfrak{A} \in CA_\alpha^+$ *if and only if* $\mathfrak{A} \in I\, D_\alpha$.

where

- CA_α^+: the system of axioms of CA_α is expanded by TWO merry-go-round axioms (MGR) and

axiom C_4 is replaced by its weakening $(\mathrm{C}_4)^* := \mathrm{d}_{ik} \cdot \mathrm{c}_i \mathrm{c}_j x \leq \mathrm{c}_j \mathrm{c}_i x$ where i, j, k and m are different.

- $\mathrm{D}_\alpha = \mathrm{Crs}_\alpha \cap \mathrm{Mod}\{\mathrm{C}_i \mathrm{D}_{ij} = V\}$.

MGR axioms:

$$\mathrm{s}_i^k \mathrm{s}_j^i \mathrm{s}_k^j\, \mathrm{c}_k x = \mathrm{s}_j^k \mathrm{s}_i^j \mathrm{s}_k^i\, \mathrm{c}_k x$$

$$\mathrm{s}_i^k \mathrm{s}_j^i \mathrm{s}_m^j \mathrm{s}_k^m\, \mathrm{c}_k x = \mathrm{s}_j^k \mathrm{s}_m^j \mathrm{s}_i^m \mathrm{s}_k^i\, \mathrm{c}_k x$$

for distinct ordinals i, j, k and n.

Remark: Andréka's short proof, Sayed Ahmed's new proof (recently)

Theorem

The axiom $(C_4)^*$ *is equivalent to* $^{-}(C_4)$, *where*

$$^{-}(\mathrm{C}_4)\colon\ s^i_k s^j_m x = s^j_m s^i_k x \quad i,k \notin \{j,m\}\,.$$

Notation: $\mathrm{CNA}^{+}_{\alpha}$ (non-commutative cylindric algebras): axiom (C_4) is replaced by $^{-}(\mathrm{C}_4)$.

Problem

What is the background (meaning) of the merry-go-round axioms?

- On transposition operators

Theorem

The axiom $(C_4)^*$ is equivalent to $^-(C_4)$, where

$^-(\mathrm{C}_4)$: $\mathrm{s}^i_k \mathrm{s}^j_m x = \mathrm{s}^j_m \mathrm{s}^i_k x \quad i, k \notin \{j, m\}$.

Notation: CNA^+_α (non-commutative cylindric algebras): axiom (C_4) is replaced by $^-(\mathrm{C}_4)$.

Problem

What is the background (meaning) of the merry-go-round axioms?

- On transposition operators
- Elementary transposition operator: $[i,\ j]$ in set algebras.

Theorem

The axiom $(C_4)^*$ is equivalent to ${}^{-}(C_4)$, where

${}^{-}(\mathrm{C}_4)$: $\mathsf{s}^i_k \mathsf{s}^j_m x = \mathsf{s}^j_m \mathsf{s}^i_k x \quad i, k \notin \{j, m\}$.

Notation: CNA^+_α (non-commutative cylindric algebras): axiom (C_4) is replaced by ${}^{-}(\mathrm{C}_4)$.

Problem

What is the background (meaning) of the merry-go-round axioms?

- On transposition operators
- Elementary transposition operator: $[i,\ j]$ in set algebras.
- Abstract transposition operator: p_{ij}.

Theorem

The axiom $(C_4)^*$ is equivalent to $^{-}(C_4)$, where

$^{-}(\mathsf{C}_4)$: $\mathsf{s}^i_k \mathsf{s}^j_m x = \mathsf{s}^j_m \mathsf{s}^i_k x \quad i,k \notin \{j,m\}$.

Notation: CNA^+_α (non-commutative cylindric algebras): axiom (C_4) is replaced by $^{-}(\mathsf{C}_4)$.

Problem

What is the background (meaning) of the merry-go-round axioms?

- On transposition operators
- Elementary transposition operator: $[i,\ j]$ in set algebras.
- Abstract transposition operator: p_{ij}.
- ${}_k s(i,\ j) = \mathsf{s}^k_i \mathsf{s}^i_j \mathsf{s}^j_k$ (i, j, k are different) is a transposition-like operator in cylindric algebras.

Theorem

The axiom $(C_4)^*$ is equivalent to ${}^{-}(C_4)$, where

${}^{-}(\mathrm{C}_4)$: $\mathrm{s}^i_k \mathrm{s}^j_m x = \mathrm{s}^j_m \mathrm{s}^i_k x \quad i, k \notin \{j, m\}$.

Notation: CNA^+_α (non-commutative cylindric algebras): axiom (C_4) is replaced by ${}^{-}(\mathrm{C}_4)$.

Problem

What is the background (meaning) of the merry-go-round axioms?

- On transposition operators
- Elementary transposition operator: $[i,\ j]$ in set algebras.
- Abstract transposition operator: p_{ij}.
- ${}_k s(i,\ j) = \mathrm{s}^k_i \mathrm{s}^i_j \mathrm{s}^j_k$ (i, j, k are different) is a transposition-like operator in cylindric algebras.
- REMARK: Not every cylindric algebra has an abstract transposition operator (Ferenczi, Sayed-Ahmed).

CONJECTURE: the meaning of MGR is the existence of a kind of transposition operator.

An equivalent form of MGR:

$$_k\mathsf{s}(i,j)\ _k\mathsf{s}(j,m)\mathsf{c}_k x = \ _k\mathsf{s}(j,m)\ _k\mathsf{s}(m,i)\mathsf{c}_k x$$

under the other CNA_α axioms if $k \notin \{i,j,m\}$, $j \notin \{m,i\}$ (Andréka-Thompson).

A variant of the Resek-Thompson-Andréka theorem:

Theorem

(Ferenczi) $\mathfrak{A} \in CNA_\alpha$ extended by $p_{ij} := \ _k s(i,j)$ is a partial transposition algebra (PTA_α) if and only if $\mathfrak{A} \in ID_\alpha$.

After the Resek-Thompson-Andréka theorem

CONJECTURE: Transposition algebras are $r-$ representable.
TA_α is a weekening of finite polyadic equality algebras (being definitionally equivalent to quasi-polyadic equality algebras) introduced by Sain and Thompson:

Definition

(TA_α) A *transposition algebra of dimension* α ($\alpha \geq 3$) is an algebra

$$\mathfrak{A} = \langle A,\ +,\ \cdot,\ -,\ 0,\ 1,\ \mathsf{c}_i,\ \ \mathsf{s}^i_j,\ \mathsf{p}_{ij},\ \ \mathsf{d}_{ij}\rangle_{i,j<\alpha},$$

where c_i, s^i_j, p_{ij} are unary operations, d_{ij} are constants, and the axioms (F0–F9) below are assumed for every $i, j, k < \alpha$:

(F0) $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, $\mathsf{s}^i_i = \mathsf{p}_{ii} = \mathsf{d}_{ii} = Id \upharpoonright A$ and $\mathsf{p}_{ij} = \mathsf{p}_{ji}$,
(F1) $x \leq \mathsf{c}_i x$,
(F2) $\mathsf{c}_i(x + y) = \mathsf{c}_i x + \mathsf{c}_i y$,
(F3) $\mathsf{s}^i_j \mathsf{c}_i x = \mathsf{c}_i x$,
(F4) $\mathsf{c}_i \mathsf{s}^i_j x = \mathsf{s}^i_j x \quad i \neq j$,
(F5)* $\mathsf{s}^i_j \mathsf{s}^k_m x = \mathsf{s}^k_m \mathsf{s}^i_j x \ \ if \ \ i, j \notin \{k, \ m\}$
(F6) s^i_j and p_{ij} are Boolean endomorphisms (i.e., $\mathsf{s}^i_j(-x) = -\ \mathsf{s}^i_j x$, etc.),
(F7) $\mathsf{p}_{ij} \mathsf{p}_{ij} x = x$,
(F8) $\mathsf{p}_{ij} \mathsf{p}_{ik} x = \mathsf{p}_{jk} \mathsf{p}_{ij} x$ if i, j, k are distinct, (MGR!)
(F9) $\mathsf{p}_{ij} \mathsf{s}^i_j x = \mathsf{s}^j_i x$,
(F10) $\mathsf{s}^i_j \mathsf{d}_{ij} = 1$,
(F11) $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x$.

Trivial: the cylindric reduct is r-representable.

Theorem

(Ferenczi) $\mathfrak{A} \in TA_\alpha$ *if and only if* $\mathfrak{A} \in I\ Gwt_\alpha$.

where the unit of a Gwt_α is of the form: $\cup_{k \in K}\ {}^{\alpha}U_k^{(pk)}$ (the difference between Gwt and Gws!)

THE GEOMETRICAL MEANING OF Gwt_α!

The reformulation of the theorem:

Theorem

The class $I\ Gwt_\alpha$ *is finite schema axiomatizable by the* TA_α *axioms.*

Stone-type theorem!

- REMARK: I Gws_α is not f.s. axiomatizable (the U_i 's are disjoint), but I Gwt_α is f.s. axiomatizable (the U_i 's are NOT disjoint)

Theorem

(Ferenczi) $\mathfrak{A} \in TA_\alpha$ *if and only if* $\mathfrak{A} \in I\ Gwt_\alpha$.

where the unit of a Gwt_α is of the form: $\cup_{k\in K}\ {}^{\alpha}U_k^{(pk)}$ (the difference between Gwt and Gws!)

THE GEOMETRICAL MEANING OF Gwt_α!

The reformulation of the theorem:

Theorem

The class $I\ Gwt_\alpha$ *is finite schema axiomatizable by the* TA_α *axioms.*

Stone-type theorem!

- REMARK: I Gws_α is not f.s. axiomatizable (the U_i 's are disjoint), but I Gwt_α is f.s. axiomatizable (the U_i 's are NOT disjoint)
- PROBLEM: Generalization for polyadic equality algebras?

Theorem

(Andréka) $\mathfrak{A} \in CNA_\alpha^+ \cap Mod\ \{Ax_{ij}\}$ *if and only if* $\mathfrak{A} \in I\ G_\alpha$, *if* α *is finite.*

where $\mathsf{Ax}_{ij} := x \leq \mathsf{c}_i\mathsf{c}_j(\mathsf{s}_j^i\mathsf{c}_j x \cdot \mathsf{s}_i^j\mathsf{c}_i x \cdot \prod_{k<\alpha, k\neq i,j} \mathsf{s}_i^k\ \mathsf{s}_j^i\ \mathsf{s}_k^j\mathsf{c}_k x) \quad i,j < \alpha.$

the unit V of a G_α is of the form $V = \cup_{k\in K}\ {}^\alpha U_k$ where the sets U_k are arbitrary (the difference between G_α and Gs_α!) - locally square algebras.

- THE GEOMETRICAL MEANING OF G_α!

Theorem

(Andréka) $\mathfrak{A} \in CNA_\alpha^+ \cap Mod\,\{Ax_{ij}\}$ *if and only if* $\mathfrak{A} \in I\,G_\alpha$, *if* α *is finite.*

where $\mathsf{Ax}_{ij} := x \leq \mathsf{c}_i\mathsf{c}_j(\mathsf{s}_j^i\mathsf{c}_j x \cdot \mathsf{s}_i^j\mathsf{c}_i x \cdot \prod_{k<\alpha, k\neq i,j} \mathsf{s}_i^k\ \mathsf{s}_j^i\ \mathsf{s}_k^j \mathsf{c}_k x)\quad i,j<\alpha.$

the unit V of a G_α is of the form $V = \cup_{k\in K}\, {}^\alpha U_k$ where the sets U_k are arbitrary (the difference between G_α and Gs_α!) - locally square algebras.

- THE GEOMETRICAL MEANING OF G_α!
- A reformulation of the theorem: The class G_α is *finite schema axiomatizable* in terms of the above schema.

Theorem

(Andréka) $\mathfrak{A} \in CNA_\alpha^+ \cap$ *Mod* $\{Ax_{ij}\}$ *if and only if* $\mathfrak{A} \in I\ G_\alpha$, *if* α *is finite.*

where $\mathsf{Ax}_{ij} := x \leq \mathsf{c}_i\mathsf{c}_j(\mathsf{s}_j^i\mathsf{c}_j x \cdot \mathsf{s}_i^j\mathsf{c}_i x \cdot \prod_{k<\alpha, k\neq i,j} \mathsf{s}_i^k\ \mathsf{s}_j^i\ \mathsf{s}_k^j\mathsf{c}_k x) \quad i,j < \alpha.$

the unit V of a G_α is of the form $V = \cup_{k\in K}\ {}^\alpha U_k$ where the sets U_k are arbitrary (the difference between G_α and Gs_α!) - locally square algebras.

- THE GEOMETRICAL MEANING OF G_α!
- A reformulation of the theorem: The class G_α is *finite schema axiomatizable* in terms of the above schema.
- PROBLEM: The generalization for infinite α?

Definition

(CPE_α) A *cylindric polyadic equality algebra* of dimension α is an algebraic structure

$$\mathfrak{A} = \langle A,\ +,\ \cdot,\ -,\ 0,\ 1,\ \mathsf{c}_i,\ \mathsf{s}_\tau,\ \mathsf{d}_{ij}\rangle_{\tau\in{}^\alpha\alpha,\ i.j\in\alpha}$$

where $+$, and $\cdot$ are binary operations on A, $-$, c_i and s_τ are unary operations on A, $0, 1$ and d_{ij} are elements of A such that for every $i, j \in \alpha$, $x, y \in A$, $\sigma, \tau \in {}^\alpha\alpha$ the usual polyadic postulates are satisfied.

Theorem

(Ferenczi) $\mathfrak{A} \in CPE_\alpha$ *if and only if* $\mathfrak{A} \in I\ Gp_\alpha^{reg}$.

where the unit of a Gp_α^{reg} is of the form: $\cup_{k\in K}\ {}^\alpha U_k$ (the difference between Gp_α^{reg} and Gs_α!)

- THE GEOMETRICAL MEANING OF Gp_α^{reg} !

Definition

(CPE_α) A *cylindric polyadic equality algebra* of dimension α is an algebraic structure

$$\mathfrak{A} = \langle A,\ +,\ \cdot,\ -,\ 0,\ 1,\ \mathsf{c}_i,\ \mathsf{s}_\tau,\ \mathsf{d}_{ij} \rangle_{\tau \in {}^\alpha\alpha,\ i.j \in \alpha}$$

where $+$, and $\cdot$ are binary operations on A, $-$, c_i and s_τ are unary operations on A, $0, 1$ and d_{ij} are elements of A such that for every $i, j \in \alpha$, $x, y \in A$, $\sigma, \tau \in {}^\alpha\alpha$ the usual polyadic postulates are satisfied.

Theorem

(Ferenczi) $\mathfrak{A} \in CPE_\alpha$ *if and only if* $\mathfrak{A} \in I\ Gp_\alpha^{reg}$.

where the unit of a Gp_α^{reg} is of the form: $\cup_{k \in K}\ {}^\alpha U_k$ (the difference between Gp_α^{reg} and Gs_α!)

- THE GEOMETRICAL MEANING OF Gp_α^{reg} !
- The reformulation of the theorem: The class I Gp_α^{reg} is *finite schema axiomatizable* by the CPE_α axioms.

On the connection of r-representation and neat embeddability.

The classical theorem:

Theorem

$\mathfrak{A} \in CA_\alpha$ is representable if and only if $\mathfrak{A} \in SNr_\alpha CA_{\alpha+\omega}$.

Generalization for $r-$representation:

Theorem

(Ferenczi) $\mathfrak{A}$ is r-representable if and only if $\mathfrak{A} \in SNr_\alpha F^\alpha_{\alpha+\omega}$,

where $\mathrm{F}^\alpha_{\alpha+\omega}$ is a many-sorted cylindric-like algebra.

- REMARK: $\mathfrak{A} \in \mathrm{SNr}_\alpha PEA_{\alpha+\omega} \not\Rightarrow \mathfrak{A}$ is representable!!! (Daigneault-Monk-Keisler)

On the connection of r-representation and neat embeddability.

The classical theorem:

Theorem

$\mathfrak{A} \in CA_\alpha$ is representable if and only if $\mathfrak{A} \in SNr_\alpha CA_{\alpha+\omega}$.

Generalization for $r-$representation:

Theorem

(Ferenczi) $\mathfrak{A}$ is r-representable if and only if $\mathfrak{A} \in SNr_\alpha F^\alpha_{\alpha+\omega}$,

where $\mathrm{F}^\alpha_{\alpha+\omega}$ is a many-sorted cylindric-like algebra.

- REMARK: $\mathfrak{A} \in \mathrm{SNr}_\alpha PEA_{\alpha+\omega} \not\Rightarrow \mathfrak{A}$ is representable!!! (Daigneault-Monk-Keisler)
- BUT: $\mathfrak{A} \in \mathrm{SNr}_\alpha PEA_{\alpha+\omega} \Longrightarrow \mathfrak{A}$ is r-representable!

To the memory of Leon Henkin