

A completeness theorem using algebraic logic

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$$\phi \vee (\psi \vee \chi) \leftrightarrow (\phi \vee \psi) \vee \chi$$

$$\phi \wedge (\psi \wedge \chi) \leftrightarrow (\phi \wedge \psi) \wedge \chi$$

$$\phi \vee \psi \leftrightarrow \psi \vee \phi$$

$$\phi \wedge \psi \leftrightarrow \psi \wedge \phi$$

$$\phi \vee (\phi \wedge \psi) \leftrightarrow \phi$$

$$\phi \wedge (\phi \vee \psi) \leftrightarrow \phi$$

$$\phi \vee (\psi \wedge \chi) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$$

$$\phi \wedge (\psi \vee \chi) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$$

$$\phi \vee \neg \phi \leftrightarrow \top$$

$$\phi \wedge \neg \phi \leftrightarrow \perp$$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

$$x \vee y = y \vee x$$

$$x \wedge y = y \wedge x$$

$$x \vee (x \wedge y) = x$$

$$x \wedge (x \vee y) = x$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee \neg x = 1$$

$$x \wedge \neg x = 0$$

$$\diamond(\phi \vee \psi) \leftrightarrow \diamond\phi \vee \diamond\psi$$

$$\diamond\perp \leftrightarrow \perp$$

$$c(x \vee y) = cx \vee cy$$

$$c0 = 0$$

Let

$$s_{ij}\phi =_{def} \diamond_i(\delta_{ij} \wedge \phi) \quad s_{ij}x =_{def} c_i(d_{ij} \wedge x)$$

$$\phi \rightarrow \diamond_i \phi$$

$$x \leq c_i x$$

$$\diamond_i \neg \diamond_i \phi \leftrightarrow \neg \diamond_i \phi$$

$$c_i \neg c_i x = \neg c_i x$$

$$\delta_{ii} \leftrightarrow \top$$

$$d_{ii} = 1$$

$$\delta_{ij} \leftrightarrow \delta_{ji}$$

$$d_{ij} = d_{ji}$$

$$\delta_{ik} \leftrightarrow s_{ji}\delta_{jk} \text{ if } j \neq i, j \neq k \quad d_{ik} = s_{ji}d_{jk} \text{ if } j \neq i, j \neq k$$

$$\delta_{ij} \wedge s_{ij}\phi \rightarrow \phi \text{ if } j \neq i \quad d_{ij} \wedge c_{ij}x \text{ if } j \neq i$$

And, for the locally finite case: for all $x \in A$ or p a propositional letter, there is a $k \in \omega$, such that for all $j \in \omega$, $j \geq k$,

$$p \leftrightarrow \diamond_j p \quad x = c_j x$$

$$\lambda \quad (\lambda \quad 1 \quad (\lambda \quad 1)) \quad (\lambda \quad 2 \quad 1)$$

The diagram illustrates the nested structure of the lambda expression $\lambda (\lambda (\lambda 1) 1) (\lambda 2 1)$. Colored arrows highlight the following connections:

- A red arrow connects the first λ to the 2 in the final argument $(\lambda 2 1)$.
- A blue arrow connects the second λ to the first 1 in the second argument $(\lambda 1 (\lambda 1 1))$.
- An orange arrow connects the third λ to the 1 in the third argument $(\lambda 1 1)$.
- A teal arrow connects the λ in the final argument $(\lambda 2 1)$ to the 1 in the same argument.

Let

$$s_i\phi =_{def} \diamond(\delta_i \wedge \phi)$$

$$DA1 : \delta_i \leftrightarrow s_1\delta_{i+1}$$

$$DA2 : s_i\delta_j \leftrightarrow \diamond s_{i+1}\delta_{j+1}$$

$$DA3 : s_i\delta_j \leftrightarrow -\diamond -s_{i+1}\delta_{j+1}$$

$$DA4 : s_i\phi \leftrightarrow -s_i - \phi$$

And, for the locally finite case: for all $j \in \omega$, $j \geq type(\phi)$,

$$LF'1 : \phi \leftrightarrow s_k^k\phi$$

$$LF'2 : s_{i_0+0} \dots s_{i_{k-1}+k-1}\phi \leftrightarrow \diamond s_{i_0+1} \dots s_{i_{k-1}+k}\phi$$

$$LF'3 : s_{i_0+0} \dots s_{i_{k-1}+k-1}\phi \leftrightarrow -\diamond -s_{i_0+1} \dots s_{i_{k-1}+k}\phi$$

$$LF'4 : s_{i_0+0} \dots s_{i_{k-2}+k-2} \diamond \phi \leftrightarrow \diamond s_{i_0+1} \dots s_{i_{k-2}+k-1} s_k \phi$$

$$\begin{aligned}
(p(v_{k-1} \dots v_0))^m &= p, \quad \text{where } k = \rho(p) \\
(v_i = v_j)^m &= s_{i+1} d_{j+1} \\
(-\psi)^m &= -(\psi^m) \\
(\psi_1 \wedge \psi_2)^m &= \psi_1^m \wedge \psi_2^m \\
(\exists v_i \psi)^m &= s_j^{j-(i+1)} c s_j^i \psi^m \\
\text{where } j &= \text{type}'(\exists v_i \psi)
\end{aligned}$$

$$\begin{aligned}
(p)^f &= p(v_{k-1}, \dots, v_0), \quad \text{where } k = \rho(p) \\
(d_i)^f &= v_0 = v_i \\
(-\phi)^f &= -(\phi^f) \\
(\phi_1 \wedge \phi_2)^f &= \phi_1^f \wedge \phi_2^f \\
(c\phi)^f &= s_{[j/j-1]} s_{[j-1/j-2]} \dots s_{[1/0]} \exists v_0 \phi^f \\
\text{where } j &= \text{type}'(\phi^f)
\end{aligned}$$

For every $t \in T$, $\phi \in Fm^m$, $\psi \in Fm^f$,

$$\begin{aligned}\mathcal{M}^m \models_t^m \phi & \text{ iff } \mathcal{M}^f \models^f \phi^f[t] \\ \mathcal{M}^f \models^f \psi[t] & \text{ iff } \mathcal{M}^m \models_t^m \psi^m\end{aligned}$$

For any $\phi \in Fm^m$,

$$\vdash^m ((\phi)^f)^m \rightarrow \phi$$

An algebra is a *set* DA (Ds) algebra if it is of the form:

$$\mathbf{S}\{\langle P(U^\omega), \cap, -, \mathsf{T}_{succ}, \mathsf{D}_{0i} \rangle_{i \in \omega^+} : U \text{ is a set}\}$$

where $\mathsf{T}_{succ}(X) = \{t \circ succ : t \in X\}$,
 $\mathsf{D}_{0i} = \{t \in U^\omega : t_0 = t_i\}$.

A set DA algebra A is *regular* if for every $x \in A$ there is a $k \in \omega$ such that:

$$\forall t, t' \in 1_A((t_{k-1} \dots t_0 = t'_{k-1} \dots t'_0) \rightarrow (t \in x \text{ iff } t' \in x))$$

Let

$$s_i x =_{def} c(d_i \wedge \phi)$$

$$DA1 : d_i = s_1 d_{i+1}$$

$$DA2 : s_i d_j = c s_{i+1} d_{j+1}$$

$$DA3 : s_i d_j = -c - s_{i+1} d_{j+1}$$

$$DA4 : s_i x = -s_i - x$$

And, for the locally finite case: for all $x \in A$, there is a $k \in \omega$, such that for all $j \in \omega$, $j \geq k$,

$$LF'1 : x = s_k^k x$$

$$LF'2 : s_{i_0+0} \dots s_{i_{k-1}+k-1} x = c s_{i_0+1} \dots s_{i_{k-1}+k} x$$

$$LF'3 : s_{i_0+0} \dots s_{i_{j-1}+k-1} x = -c - s_{i_0+1} \dots s_{i_{k-1}+k} x$$

$$LF'4 : s_{i_0+0} \dots s_{i_{k-2}+k-2} c x = c s_{i_0+1} \dots s_{i_{k-2}+k-1} s_k x$$

Every simple countable locally finite DA is isomorphic to a regular set DA.

The proof is analogous to the representation proof in AN75.