A completeness theorem using algebraic logic

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$$\phi \vee (\psi \vee \chi) \leftrightarrow (\phi \vee \psi) \vee \chi \qquad x \vee (y \vee z) = (x \vee y) \vee z$$
$$\phi \wedge (\psi \wedge \chi) \leftrightarrow (\phi \wedge \psi) \wedge \chi \qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z$$
$$\phi \vee \psi \leftrightarrow \psi \vee \phi \qquad x \vee y = y \vee x$$
$$\phi \wedge \psi \leftrightarrow \psi \wedge \phi \qquad x \wedge y = y \wedge x$$
$$\phi \vee (\phi \wedge \psi) \leftrightarrow \phi \qquad x \vee (x \wedge y) = x$$
$$\phi \wedge (\phi \vee \psi) \leftrightarrow \phi \qquad x \wedge (x \vee y) = x$$
$$\phi \vee (\psi \wedge \chi) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$\phi \wedge (\psi \vee \chi) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$\phi \vee \neg\phi \leftrightarrow \top \qquad x \vee \neg x = 1$$
$$\phi \wedge \neg\phi \leftrightarrow \bot \qquad x \wedge \neg x = 0$$

$$\diamond(\phi \vee \psi) \leftrightarrow \diamond\phi \vee \diamond\psi \qquad c(x \vee y) = cx \vee cy$$
$$\diamond\bot \leftrightarrow \bot \qquad c0 = 0$$

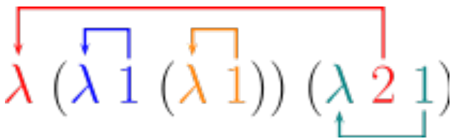
Let

$$s_{ij}\phi =_{def} \diamond_i(\delta_{ij} \wedge \phi) \;\; s_{ij}x =_{def} c_i(d_{ij} \wedge x)$$

$\phi \to \diamond_i\phi$	$x \le c_i x$
$\diamond_i \neg \diamond_i \phi \leftrightarrow \neg \diamond_i \phi$	$c_i \neg c_i x = \neg c_i x$
$\delta_{ii} \leftrightarrow \top$	$d_{ii} = 1$
$\delta_{ij} \leftrightarrow \delta_{ji}$	$d_{ij} = d_{ji}$
$\delta_{ik} \leftrightarrow s_{ji}\delta_{jk}$ if $j \neq i, j \neq k$	$d_{ik} = s_{ji}d_{jk}$ if $j \neq i, j \neq k$
$\delta_{ij} \wedge s_{ij}\phi \to \phi$ if $j \neq i$	$d_{ij} \wedge c_{ij}x$ if $j \neq i$

And, for the locally finite case: for all $x \in A$ or p a propositional letter, there is a $k \in \omega$, such that for all $j \in \omega$, $j \geq k$,

$$p \leftrightarrow \diamond_j p \;\; x = c_j x$$


λ (λ 1 (λ 1)) (λ 2 1)

Let

$$s_i\phi =_{def} \diamond(\delta_i \wedge \phi)$$

$$\begin{aligned} DA1 &: \ \delta_i \ \leftrightarrow \ s_1\delta_{i+1} \\ DA2 &: \ s_i\delta_j \ \leftrightarrow \ \diamond s_{i+1}\delta_{j+1} \\ DA3 &: \ s_i\delta_j \ \leftrightarrow \ -\diamond - s_{i+1}\delta_{j+1} \\ DA4 &: \ s_i\phi \ \leftrightarrow \ -s_i - \phi \end{aligned}$$

And, for the locally finite case: for all $j \in \omega$, $j \geq type(\phi)$,

$$\begin{aligned} LF'1 &: \ \phi \ \leftrightarrow \ s_k^k\phi \\ LF'2 &: \ s_{i_0+0} \dots s_{i_{k-1}+k-1}\phi \ \leftrightarrow \ \diamond s_{i_0+1} \dots s_{i_{k-1}+k}\phi \\ LF'3 &: \ s_{i_0+0} \dots s_{i_{k-1}+k-1}\phi \ \leftrightarrow \ -\diamond - s_{i_0+1} \dots s_{i_{k-1}+k}\phi \\ LF'4 &: \ s_{i_0+0} \dots s_{i_{k-2}+k-2} \diamond \phi \ \leftrightarrow \ \diamond s_{i_0+1} \dots s_{i_{k-2}+k-1}s_k\phi \end{aligned}$$

$(p(v_{k-1} \dots v_0))^m = p,$ where $k = \rho(p)$
$(v_i = v_j)^m = s_{i+1} d_{j+1}$
$(-\psi)^m = -(\psi^m)$
$(\psi_1 \wedge \psi_2)^m = \psi_1^m \wedge \psi_2^m$
$(\exists v_i \psi)^m = s_j^{j-(i+1)} c s_j^i \psi^m$
where $j = type'(\exists v_i \psi)$

$(p)^f = p(v_{k-1}, \dots, v_0),$ where $k = \rho(p)$
$(d_i)^f = v_0 = v_i$
$(-\phi)^f = -(\phi^f)$
$(\phi_1 \wedge \phi_2)^f = \phi_1^f \wedge \phi_2^f$
$(c\phi)^f = s_{[j/j-1]} s_{[j-1/j-2]} \dots s_{[1/0]} \exists v_0 \phi^f$
where $j = type'(\phi^f)$

For every $t \in T$, $\phi \in Fm^m$, $\psi \in Fm^f$,

$$\mathcal{M}^m \models_t^m \phi \quad \text{iff} \quad \mathcal{M}^f \models^f \phi^f[t]$$
$$\mathcal{M}^f \models^f \psi[t] \quad \text{iff} \quad \mathcal{M}^m \models_t^m \psi^m$$

For any $\phi \in Fm^m$,

$$\vdash^m ((\phi)^f)^m \rightarrow \phi$$

An algebra is a *set* DA (Ds) algebra if it is of the form:

$$\mathbf{S}\{\langle P(U^\omega), \cap, -, \mathsf{T}_{succ}, \mathsf{D}_{0i}\rangle_{i\in\omega^+} : U \text{ is a set}\}$$

where $\mathsf{T}_{succ}(X) = \{t \circ succ : t \in X\}$,
$\mathsf{D}_{0i} = \{t \in U^\omega : t_0 = t_i\}$.

A set DA algebra A is *regular* if for every $x \in A$ there is a $k \in \omega$ such that:

$$\forall t, t' \in 1_A((t_{k-1} \dots t_0 = t'_{k-1} \dots t'_0) \to (t \in x \text{ iff } t' \in x))$$

Let

$$s_i x =_{def} c(d_i \wedge \phi)$$

$$\begin{aligned} DA1 &: \ d_i = s_1 d_{i+1} \\ DA2 &: \ s_i d_j = c s_{i+1} d_{j+1} \\ DA3 &: \ s_i d_j = -c - s_{i+1} d_{j+1} \\ DA4 &: \ s_i x = -s_i - x \end{aligned}$$

And, for the locally finite case: for all $x \in A$, there is a $k \in \omega$, such that for all $j \in \omega$, $j \geq k$,

$$\begin{aligned} LF'1 &: \ x = s_k^k x \\ LF'2 &: \ s_{i_0+0} \ldots s_{i_{k-1}+k-1} x = c s_{i_0+1} \ldots s_{i_{k-1}+k} x \\ LF'3 &: \ s_{i_0+0} \ldots s_{i_{j-1}+k-1} x = -c - s_{i_0+1} \ldots s_{i_{k-1}+k} x \\ LF'4 &: \ s_{i_0+0} \ldots s_{i_{k-2}+k-2} c x = c s_{i_0+1} \ldots s_{i_{k-2}+k-1} s_k x \end{aligned}$$

Every simple countable locally finite DA is isomorphic to a regular set DA.

The proof is analogous to the representation proof in AN75.