Constructibility and Space-Time

David J. BenDaniel
Cornell University
Ithaca, New York 14853

Principal Results

- Space-time is developed from a constructible set theoretical foundation
- The real line is countable and functions of a real variable are locally homeomorphic with the real line
- Space-time is relational and its differential properties fulfill the requirements of Einstein-Weyl causality
- The Schrödinger equation is developed in this theory
- This result also infers that quantum mechanics in this relational space-time framework can be considered conceptually cumulative with prior physics.

Agenda

(1) Axioms of a constructible set theory are presented.

(2) Continuous and differentiable functions of a real variable are developed in this theory.

(3) A nonlinear sigma model is postulated and the Schrödinger equation is shown to be a special case.

(4) Implications of the theory to space-time are then presented.

(5) Other examples are given of questions in physics that now may be answered by this approach.

ZF – Axiom Schema of Subsets – Power Set + Constructibility

Extensionality	Two sets with just the same members are equal.
Pairs	For every two sets, there is a set that contains just them.
Union	For every set of sets, there is a set with just all their members.
Infinity	There is at least one set ω* which contains the null set and for every member there is a next member that contains just all its predecessors
Arithmetic	The Peano axioms without the induction axiom
Regularity	Every non-empty set has a minimal member (i.e. “weak” regularity).
Replacement	Replacing members of a set one-for-one creates a set. (i.e. bijective replacement)
Constructibility	Subsets of ω* are countably constructible.

The power set axiom and the axiom schema of subsets are deleted and an axiom of constructibility adjoined.

Then:

- The theory is uniformly dependent on ω^*
- Both finite and infinite natural numbers exist in ω^*
- The set of constructible subsets of ω^* is countable
- All sets of finite natural numbers are finite
- Convergence of “infinite series” is undecidable.

The set of all ratios of any two elements of ω^* is called Q^*.

Q^* is an enlargement of the usual set of rational numbers Q.

Two members of Q^* are called "identical" if their ratio is 1.

We can now employ the symbol "$\equiv$" for "is identical to".

An “infinitesimal” is a member of Q* “equal” to 0, i.e., letting y signify the member and employing the symbol “=” to signify equality, $y = 0 \leftrightarrow \forall n[y < 1/n]$ where n is an integer.

The reciprocal of an infinitesimal is “infinite”. A member of Q* that is not an infinitesimal and not infinite is “finite”.

Note that $x \equiv y \rightarrow x = y$ but not the converse.

The constructibility axiom well-orders the set of constructible subsets of ω^*, creating a metric space. These subsets are binimals making up a real line **R***.

-

-

An *equality-preserving* bijective map $\Phi(x,u)$ between intervals X and U of **R*** in which $x \in X$ and $u \in U$

$$\forall x_1, x_2, u_1, u_2 \left[\Phi(x_1,u_1) \wedge \Phi(x_2,u_2) \rightarrow [x_1 - x_2 = 0 \leftrightarrow u_1 - u_2 = 0]\right]$$

creates pieces biunique and homeomorphic with **R***.

Note that U can be infinitesimal if and only if X is, i.e, the range vanishes if and only if the domain vanishes.

$u(x)$ is a function of a real variable in this theory if and only if it is a constant or a sequence in x of continuously connected biunique pieces such that the derivative with respect to x is also a function of a real variable. These functions are of bounded variation, locally homeomorphic with the real line **R*** and, if not constant, the range of $u(x)$ is finite (range $u(x) \neq 0$) if and only if its domain is finite.

If some derivative is a constant, these are polynomials. If no derivative is a constant, these functions do not exist in this theory. They can, however, be approached as closely as necessary by a sum of polynomials of sufficiently high degree obtained by many iterations of the following algorithm, thereby creating effectively a finite dimensional Hilbert space:

$$\int_a^b [p\left(\frac{du}{dx}\right)^2 - qu^2]dx \equiv \lambda; \quad \int_a^b ru^2 dx \equiv Const.$$

where $a \neq b$; $u\frac{du}{dx} \equiv 0$ at a and b; p, q and r are functions of x.

Polynomials of sufficiently high degree are obtained by iteratively minimizing λ for a given p, q and r. They are effectively Sturm-Liouville eigenfunctions and are of bounded variation and locally homeomorphic with R*.

This has both physical and philosophical implications.

Consider a fixed string of finite length:

$$\int \left[\left(\frac{\partial u_x u_t}{\partial x} \right)^2 - \left(\frac{\partial u_x u_t}{\partial t} \right)^2 \right] dx\, dt \equiv 0$$

The u_x and u_t can be effectively obtained by iteration subject to the constraint $\lambda_x - \lambda_t \equiv 0$

This can be generalized to more complex fields and to finitely many dimensions of a compactified space-time.

Let $u_{\ell m i}(x_i)$ and $u_{\ell m j}(x_j)$ be eigenfunctions with positive eigenvalues $\lambda_{\ell m i}$ and $\lambda_{\ell m j}$ respectively.

We now define a "field" as a sum of eigenstates

$$\underline{\Psi}_m = \sum_{\ell} \Psi_{\ell m} \underline{i_\ell}, \Psi_{\ell m} = \prod_{\mathrm{i}} \mathrm{u}_{\ell m i} \prod_{\mathrm{j}} \mathrm{u}_{\ell m j}$$

subject to the novel postulate that for every eigenstate *m* the value of the integral of the Lagrange density over ***dsdτ***, where

$dsd\tau = \prod_i r_i dx_i \prod_j r_j dx_j$, is ***identically null***: For all *m*

$$\sum_{\ell} \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell m i} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell m i} \Psi_{\ell m}^2 \right] - \sum_j \frac{1}{r_j} \left[P_{\ell m j} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell m j} \Psi_{\ell m}^2 \right] \right\} dsd\tau \equiv 0$$

over a compactified space-time; this is a **nonlinear sigma model**

.

The postulate asserts that the two symmetric components of the nonlinear sigma model have identical magnitudes.

In this nonlinear sigma model the P and Q can be functions of any of the χ_i and χ_j,

thus of any $\Psi_{\ell m}$ as well, subject to the condition that all the eigenvalues are finite and positive.

The Ψ_m are effectively given by iteration,

subject to the constraint

$$\sum_{\ell}\left(\sum_{i}\lambda_{\ell mi}-\sum_{j}\lambda_{\ell mj}\right)\equiv 0$$

Let both $$\sum_m \sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} ds\, d\tau$$ and

$$\sum_m \sum_\ell \int \left\{ \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} ds\, d\tau$$ be represented by α(Ψ)

I. α(Ψ) is positive and closed to addition and to the absolute value of subtraction, so it is either continuous or discrete.

II. α(Ψ) ≡ 0 ↔ Ψ ≡ 0; otherwise, as Ψ is a function of real variables, the range Ψ ≠ 0 and ∀Ψ α(Ψ) ≠ 0, thus α(Ψ) is not continuous.

III. ∴ α(Ψ) is discrete; α(Ψ) ≡ $n\kappa$, where n is any integer and κ is some finite unit which must be obtained empirically.

Let $\ell = 1, 2$, $r_t = P_{1mt} = P_{2mt} = 1$, $Q_{1mt} = Q_{2mt} = 0$, $\tau = \omega_m t$ and we normalize Ψ as follows:

$$\Psi_m = \sqrt{(C/2\pi)} \prod_i u_{im}(x_i)[u_{1m}(\tau) + i \cdot u_{2m}(\tau)] \quad (9)$$

where $i = \sqrt{-1}$ with

$$\int \sum_m \prod_i u_{im}^2 ds (u_{1m}^2 + u_{2m}^2) \equiv 1 \quad (10)$$

then:

$$\frac{du_{1m}}{d\tau} = -u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = u_{1m} \quad (11)$$

or

$$\frac{du_{1m}}{d\tau} = u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = -u_{1m} \quad (12)$$

For the minimal non-vanishing field, α is just the least finite value κ. Thus,

$$(C/2\pi) \sum_m \oint \int \left[\left(\frac{du_{1m}}{d\tau} \right)^2 + \left(\frac{du_{2m}}{d\tau} \right)^2 \right] \prod_i u_{im}^2(x_i) ds d\tau \equiv C \equiv \kappa \quad (13)$$

Substituting the Planck constant h for κ, this can now be put into the conventional Lagrangian form for the time term in the Schrödinger equation,

$$\frac{h}{2i} \sum_m \oint \int \left[\Psi_m^* \left(\frac{\partial \Psi_m}{\partial t} \right) - \left(\frac{\partial \Psi_m^*}{\partial t} \right) \Psi_m \right] ds dt$$

Going back to the statement $\forall\Psi\ \alpha(\Psi) \neq 0$, we must recognize we have assumed that space-time exists, i.e., that the upper and lower limits of at least one of the multiple integrals over all space-time are not equal. Otherwise, space-time does not exist $\rightarrow \forall\Psi\ \alpha(\Psi) = 0$ thus we obtain a necessary and sufficient condition:

space-time exists $\leftrightarrow \exists\Psi\ \alpha(\Psi) \neq 0$

This is a description of a relational space-time.

Furthermore, it has been shown by others that Q^2 is an ordered space that fulfills the strict Hausdorff topology requirements for Einstein-Weyl causality. Also, if Q^2 can be embedded in R^2 so that fields are locally homeomorphic with R, this will even apply to R^n (n is any positive integer).

In this theory, Q^{*2} is an ordered space one-to-one with R^{*2}, fields are by definition locally homeomorphic with R* and so R^{*n} fulfills the requirements for Einstein-Weyl causality.

For Further Discussion

- This theory proposes that the physical universe is composed entirely of constructible sets; accordingly, the universe will not have non-constructible sets which can lead to physical antimonies. This has to be intuitively satisfying since, were there any physical antinomies, the universe would tear itself apart.
- Dyson's well-known problem, that the QED perturbation series cannot converge and thus diverges, does not apply in this theory. Instead, convergence of the perturbation series is undecidable. This provides a solution and opportunity for new physical ideas.
- Wigner's philosophical question concerning the unreasonable effectiveness of mathematics in physics may be answered directly; this effectiveness comes from their foundations being linked.

References

This is a partial list:

Gödel, K., The consistency of the axiom of choice and of the generalized continuum hypothesis. *Annals of Math. Studies*, 1940.

Cohen, P. J., Set Theory and the Continuum Hypothesis. W.A. Benjamin, New York, 1966

Gottfried, K., Inferring the statistical interpretation of quantum mechanics from the classical limit. *Nature*, 2000, 405.

Borchers, H.J., Sen, R.N., Mathematical Implications of Einstein-Weyl Causality, Springer, Berlin Heidelberg, 2006

Dyson, F. J., Divergence of perturbation theory in quantum electrodynamics. *Phys.Rev*., 1952, 85.

Wigner, E. P., The unreasonable effectiveness of mathematics in the natural sciences. *Comm. Pure and Appl. Math.* 1960, 13.

ZF – Axiom Schema of Subsets – Power Set + Constructibility = T

Extensionality - Two sets with just the same members are equal

$$\forall x \forall y [\forall z [z \in x \leftrightarrow z \in y] \rightarrow x = y]$$

Pairs - For every two sets, there is a set that contains just them.

$$\forall x \forall y \exists z [\forall w w \in z \leftrightarrow w = x \vee w = y]$$

Union - For every set of sets, there is a set with just all their members.

$$\forall x \exists y \forall z [z \in y \leftrightarrow \exists u [z \in u \wedge u \in x]]$$

Infinity - A set ω* contains 0 and every member has a successor containing just all its predecessors.

$$\exists \omega^* [0 \in \omega^* \wedge \forall x [x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*]]$$

Replacement - Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement).

Let $\phi(x,y)$ a formula in which x and y are free,

$$\forall z \forall x \in z \exists y [\phi(x,y) \wedge \forall u \in z \forall v [\phi(u,v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t [t \in r \leftrightarrow \exists s \in z \phi(s,t)]$$

Regularity - Every non-empty set has a minimal member (i.e. "weak" regularity).

$$\forall x [\exists y y \in x \rightarrow \exists y [y \in x \wedge \forall z \neg [z \in x \wedge z \in y]]]$$

Constructibility - The subsets of ω* are countably constructible.

$$\forall \omega^* \exists S [(\omega^*, 0) \in S \wedge \forall y \forall z [(y,z) \in S \rightarrow$$
$$[[\exists m_y [m_y \in y \wedge \forall v \neg [v \in y \wedge v \in m_y]] \leftrightarrow [\exists t_y \forall u [u \in t_y \leftrightarrow u \in y \wedge u \neq m_y]] \wedge (t_y \cup m_y, z \cup \{z\}) \in S]]]].$$

Arithmetic:

The four formulae (a) to (d) below (Peano arithmetic axioms) can be consistently added to T.

We use x' for xU{x}.

a) $\forall a\ a' \neq O$

b) $\forall x \forall y\ (x' = y' \rightarrow x = y)$

Let x,y,a,b be members of ω*and [x,y] and [[x,y],z] represent ordered pairs.

c) $\exists A \forall x \forall y\ E!z\ [[x, O],x] \in A \wedge [[x,y],z] \in A \rightarrow [[x,y'],z'] \in A$; this is addition: $x + y = z$

d) $\exists M \forall x \forall y\ E!z\ [[x, O],O] \in M \wedge [[x,y],z] \in M \rightarrow [[x,y'],z+x] \in M$; this is multiplication: $x \bullet y = z$

Define $[a,b]_r$ such that $[a_1,b]_r + [a_2,b]_r \equiv [a_1+a_2,b]_r$ and $[a_1,b_1]_r \equiv [a_2,b_2]_r \leftrightarrow a_1 \bullet b_2 \equiv a_2 \bullet b_1$.
The extended set of rationals Q* is the set of such pairs for all a and b in ω*.

Theorem

$\forall \omega^* \forall x(\forall\ y_1 \forall\ y_2 \forall\ z_1 \forall z_2\ ([[x,y_1],z_1] \in B \wedge [[x,y_2],z_2] \in B \rightarrow ((z_1 = z_2) \leftrightarrow (y_1 = y_2))) \rightarrow$
$\exists t(x)\ \forall z(z \in t(x) \leftrightarrow \exists y(B(x,y,z)))$, where B represents A or M.