Reducing Predicate Logic to Propositional Logic

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This is joint work with István Németi, and a direct continuation of a research Tarski began.

FOL = "First-Order Logic",
CPC = "Classical Propositional Calculus".

What is this research about?

FOL is a fantastic tool for saying what we want clearly and explicitly. We wrote axiom systems for Relativity Theory in this language. It is a central tool in mathematics.

We want to see why FOL works by trying to see its simplest version/fragment that has the same expressive power.

Insights that we gained via this research:

- A simple logic, almost CPC, which is adequate for doing all of mathematics in it,
- in which set theory (and mathematics) can be formalized,
- which has Gödel's incompleteness property,
- complexity issues concerning FOL, e.g., what small resources in FOL are enough already to formalize a Gödel sentence, what is the Turing-degree of deciding validity of a FOL-sentence,
- completeness and incompleteness theorems for the finite-variable fragments of FOL,
- intuition for the Gödel-Bernays finitely axiomatized set theory,
- solutions for long-standing problems of Tarski.

We are interested in the one-and-only logic FOL that is dear to our hearts.

We will investigate FOL by splitting it to smaller fragments, seeing their connections with each other, ... soon we are in the realm of Universal Logic Jean-Yves Beziau talked about.

In UL we are dealing with many logics, there is a definition for "logics in general", for "reducing one logic to another", there is a definition for what logics we call "propositional", etc.

Can FOL be translated to CPC? The answer is affirmative:

Theorem 1 (Jerabek 2011)

FOL can be conservatively translated to CPC, i.e., there is a conservative translation function tr *from FOL to* CPC, *which means that for all sets $Th \cup \{\varphi\}$ of FOL sentences we have*
$Th \vdash \varphi$ *if and only if* $\text{tr}(Th) \vdash \text{tr}(\varphi)$.

Moreover, EVERY countable logic can be conservatively translated to CPC.

Additional properties are needed for a translation to be useful.
Pluralist attitude of Michele Friend.

Research direction for Universal Logic:

What properties of a translation function enables it to transport what properties of the logics?

$$\text{Logic}_1 \longrightarrow \text{Logic}_2$$

FOL is a semantic logic, it has a
semantic consequence $\models$ specified by a class of models, and a
syntactic consequence $\vdash$ specified by axioms and rules.

In FOL these two coincide (completeness theorem).

In fragments of FOL they may not coincide. In the finite-variable fragments they do not coincide. (J. D. Monk's theorem)

CPC is a small fragment of FOL, namely we discard quantifiers, equality and substitutions (of one variable in place of another).

$CPC^+ = \langle F, \vdash_c \rangle$ denotes a slight enrichment of the CPC-fragment of FOL, namely, CPC^+ consists of adding three quantifiers to CPC, but still no equality and no substitutions; we take the syntactic consequence. This is the equality- and substitution-free 3-variable syntactic fragment of FOL.

Equivalent definitions for the same CPC^+ come after the theorem.

Theorem 2 (AN2011)

FOL can be reduced to propositional logic CPC$^+$*, i.e., there is a computable Boolean- and semantics-preserving translation function* tr *from FOL to* CPC$^+$*, which means that (i)-(iii) below hold for all sets* $Th \cup \{\varphi, \psi\}$ *of FOL sentences.*

(i) $Th \vdash \varphi$ *if and only if* $\mathrm{tr}(Th) \vdash_c \mathrm{tr}(\varphi)$

(ii) $Th \models \varphi$ *if and only if* $\mathrm{tr}(Th) \models \mathrm{tr}(\varphi)$

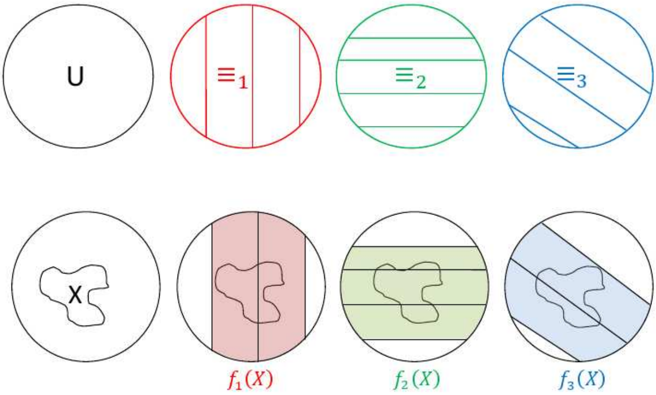
(iii) $\vdash_c \mathrm{tr}(\varphi \vee \psi) \leftrightarrow (\mathrm{tr}(\varphi) \vee \mathrm{tr}(\psi))$ *and*

$$\vdash_c \mathrm{tr}(\varphi \to \psi) \to (\mathrm{tr}(\varphi) \to \mathrm{tr}(\psi)).$$

CPC, BA, Bs,

CPC^+, BA^+, Bs^+.

An enriched Boolean set algebra is a Boolean set algebra together with three closure operations on it, specified by three commuting equivalence relations.



Bs$^+$ denotes the class of all algebras isomorphic to an enriched Boolean set algebra.

BA^+ denotes the class of Boolean algebras with 3 commuting complemented closure operators c_i ($i < 3$), i.e., c_i are unary functions on the BA satisfying the following equations for all $i, j < 3$:

$c_i c_j x = c_j c_i x$ (commuting),

$c_i - c_i x = -c_i x$ (complemented),

$x \leq c_i x = c_i c_i x$ (closure),

$c_i(x + y) = c_i x + c_i y$ (operators).

A different name for BA^+ is Df_3 for "3-dimensional diagonal-free cylindric algebras".

Extended Stone representation theorem:

Theorem 3

$BA^+ = Bs^+$.

This is the completeness theorem of modal logic $[S5, S5, S5]$.

The language of $[S5, S5, S5]$ contains infinitely many propositional variables, the connectives are $\vee, \neg, \Diamond_1, \Diamond_2, \Diamond_3$. We use $\Box_i \stackrel{d}{=} \neg\Diamond_i\neg, \rightarrow, \leftrightarrow$ as derived connectives as usual.

The axioms are the following (where φ, ψ are arbitrary formulas of the language and $i, j \in \{1, 2, 3\}$):

((B)) φ, if φ is a propositional tautology,
((K)) $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$,
((S5)) $\Diamond_i\varphi \rightarrow \Box_i\Diamond_i\varphi$,
((C1)) $\Diamond_i\Diamond_j\varphi \rightarrow \Diamond_j\Diamond_i\varphi$,
((C2)) $\Diamond_i\Box_j\varphi \rightarrow \Box_j\Diamond_i\varphi$.

The rules are Modus Ponens and Necessitation.

BA$^+$, Bs$^+$, Df_3, $[S5, S5, S5]$, "3-variable equality- and substitution-free syntactic fragment of FOL" are all different forms of CPC$^+$.

Thm.2 is optimal in that we cannot get much closer to CPC than CPC$^+$:

- 2 does not suffice in place of 3, Leon Henkin. But: upcoming talk of Agi Kurucz.
- "commuting" cannot be omitted, Németi 1985.
- We do not know whether "complemented" can be omitted or not.

Tarski 1953, reduction of Set Theory to the relational algebraic fragment of FOL,

Németi 1985, reduction of Set Theory to the 3-variable fragment of FOL,

Zalán Gyenis 2010, reduction of Set Theory to the equality-free 3-variable fragment (with substitution) of FOL,

AN 2011, reduction of FOL to the equality- and substitution-free 3-variable fragment of FOL.

Thank you for your attention!