Tutorial on Axiomatization of Relativity Theory (part 1)

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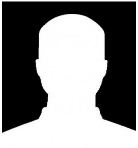

Etc.

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- Base relativity theories on simple, unambiguous axioms.
- Turn relativity theories to theories of mathematical logic.
- Analyze the logical structure of relativity theories.
- Make relativity theories modular, easier to change and extend.
- Etc.

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- Which axioms are responsible for a certain theorem?
- How are the possible axioms/axiomatizations related to each other?
- How are the independent statements of our axiomatizations related to each other?
- How can these axiomatizations be extended, e.g., towards Quantum Theory?
- Etc.

Axiomatization in general:

Axioms:		Theorems:
Ax.1.	→	Thm.1.
Ax.2.		Thm.2.
Ax.3.		Thm.3.
Etc.		Etc.

Economical
Streamlined
Transparent

Rich
Complex

We turn relativity theories to theories of FOL:

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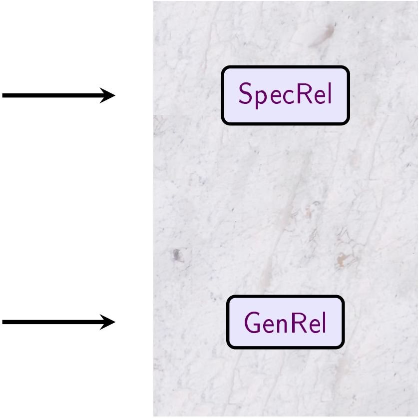

We turn relativity theories to theories of FOL:

Relativity theory is axiomatic (in its spirit) since its birth.

Two informal postulates of Einstein (1905):

- **Principle of relativity:** "The laws of nature are the same for every inertial observer."
- **Light postulate:** "Any ray of light moves in the 'stationary' system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body,"

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To axiomatize relativity theory within logic, we need a language (set of basic concepts).

Language: $\{$ B, Q, $\}$

B ⇝ Bodies (things that move)

Q ⇝ Quantities

Language: $\{ B, IOb, Ph, Q, \qquad \}$

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IOb ⇝ Inertial Observers Ph ⇝ Photons (light signals)

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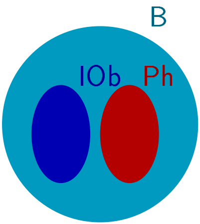
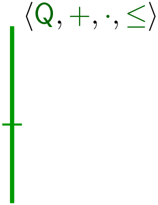
Language: $\{\, \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \quad \}$





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Q ⇝ Quantities $+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

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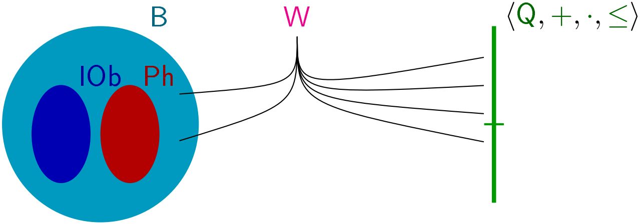

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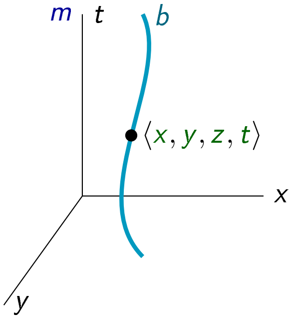
W ⇝ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, x, y, z, t) \leftrightsquigarrow$ " observer m coordinatizes body b at spacetime location $\langle x, y, z, t\rangle$."



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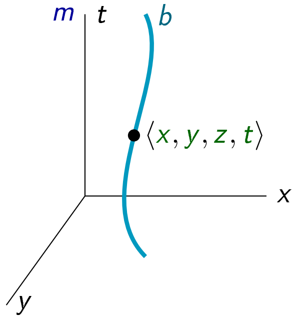
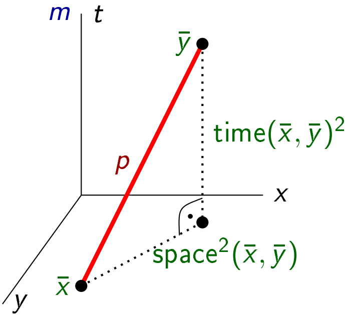


Worldline of body b according to observer m

$$wline_m(b) := \{\langle x, y, z, t \rangle \in \mathrm{Q}^4 \;:\; \mathsf{W}(m, b, x, y, z, t)\}$$

AxLight :

There is an inertial observer, according to whom, any light signal moves with the same velocity c in every direction.



$$\exists mc\Big[\mathsf{IOb}(m) \land c > 0 \land \forall \bar{x}\bar{y}\Big(\exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow \mathsf{space}^2(\bar{x},\bar{y}) = c^2 \cdot \mathsf{time}(\bar{x},\bar{y})^2\Big)\Big]$$

Let $\mathcal{F}$ be the set of **potential laws of physics**.

$\mathsf{SPR}_{\mathcal{F}}$:

Every $\varphi \in \mathcal{F}$ potential law of physics either holds for every inertial observer or none of them.

$$\{\, \mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \rightarrow [\varphi(m, \bar{x}) \leftrightarrow \varphi(k, \bar{x})] \ : \ \varphi \in \mathcal{F} \,\}.$$

Potential laws of physics $(\mathcal{F}) = ???$

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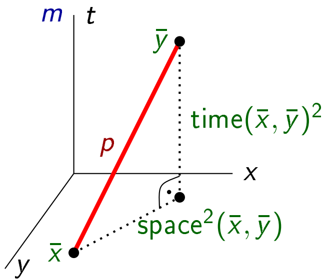
SPR^{+}: when $\mathcal{F}$ contains all the formulas having only 1 free variable of type bodies.

AxPh :

For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.

$$\mathsf{IOb}(m) \to \exists c\Big[c > 0 \land \forall \bar{x}\bar{y}\ \Big(\exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow \mathsf{space}^2(\bar{x},\bar{y}) = c^2 \cdot \mathsf{time}(\bar{x},\bar{y})^2\Big)\Big]$$



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See Attila Molnár's talk for capturing the notions of possibility in relativity within a modal logic frame.

AxPh :

For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.

Proposition.:

$$\mathsf{SPR}^{+} + \mathsf{AxLight} \Rightarrow \mathsf{AxPh}$$

$$\mathsf{SPR}_{\mathcal{F}} + \mathsf{AxLight} \Rightarrow \mathsf{AxPh}, \ \textit{if}$$

$$\exists p\big[\mathsf{Ph}(p) \wedge \mathsf{W}(m,p,\bar{x}) \wedge \mathsf{W}(m,p,\bar{y})\big] \in \mathcal{F}.$$

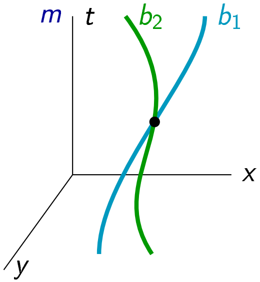
AxOField :

The **structure of quantities** $\langle Q, +, \cdot, \leq \rangle$ *is an ordered field,*

- Rational numbers: $\mathbb{Q}$,
- $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\pi)$, ...
- Computable numbers,
- Constructable numbers,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- **Real numbers**: $\mathbb{R}$,
- Hyperrational numbers: $\mathbb{Q}^*$,
- Hyperreal numbers: $\mathbb{R}^*$,
- Etc.

AxEv :

Inertial observers coordinatize the same events (meetings of bodies).



$$\forall m\, m' \bar{x} \;\; \mathsf{IOb}(m) \wedge \mathsf{IOb}(m') \rightarrow \left[\exists \bar{x}' \;\; \forall b \;\; \mathsf{W}(m, b, \bar{x}) \leftrightarrow \mathsf{W}(m', b, \bar{x}')\right].$$

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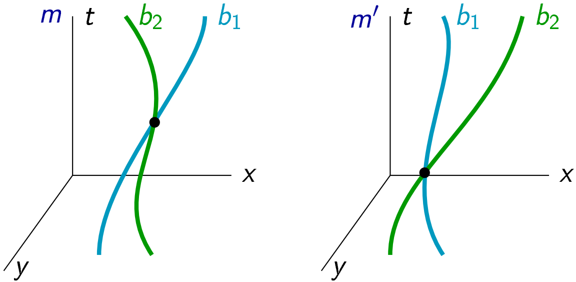
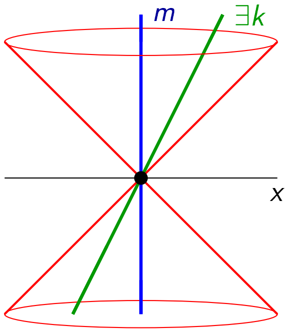


$$\forall m\, m' \bar{x}\ \ \mathsf{IOb}(m) \wedge \mathsf{IOb}(m') \to \big[\exists \bar{x}'\ \ \forall b\ \ \mathsf{W}(m,b,\bar{x}) \leftrightarrow \mathsf{W}(m',b,\bar{x}')\big].$$

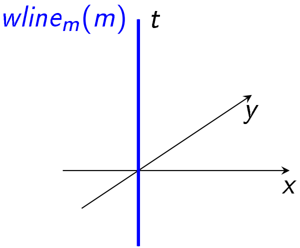
AxThExp :

Inertial observers can move with any speed slower than that of light.



AxSelf :

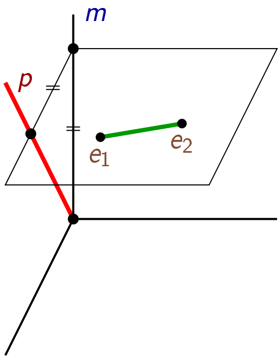
Every Inertial observer is stationary according to himself.

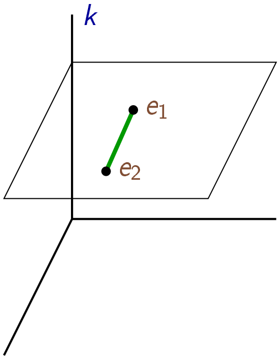


$$\forall mxyzt\ \big(\mathsf{IOb}(m) \rightarrow [\mathsf{W}(m,m,x,y,z,t) \leftrightarrow x = y = z = 0]\big).$$

AxSym :

Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them. Furthermore, the speed of light is 1.





SR:=AxOField + **SPR**$^+$ + **AxLight** + AxEv + AxThExp + AxSelf + AxSym

SpecRel:=AxOField + **AxPh** + AxEv + AxSelf + AxSym

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Proposition.:

SR $\Rightarrow$ Einstein's Special Relativity $\Rightarrow$ SpecRel

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SpecRel:=AxOField + **AxPh** + AxEv + AxSelf + AxSym

Proposition.:

SR ⇒ Einstein's Special Relativity ⇒ SpecRel, *but*
SR $\not\Leftarrow$ SpecRel.

What follows from SpecRel?

SpecRel:

AxPh
AxEv
AxOField
AxSelf
AxSym

→

Theorems:

?
??
???
Etc.

Theorems of SpecRel

SpecRel:=AxOField + **AxPh** + AxEv + AxSelf + AxSym

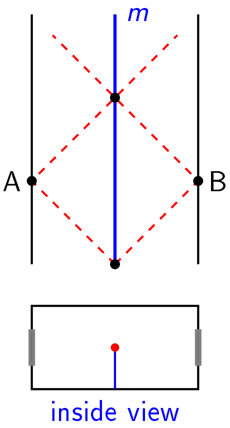
Theorem:

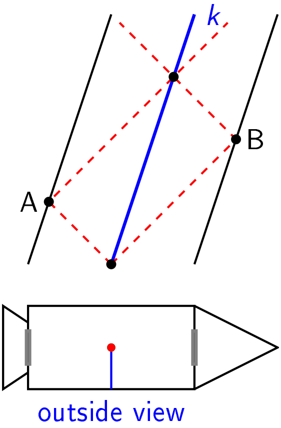
SpecRel $\vdash$ *"Worldlines of inertial observers are straight lines."*

Theorems of SpecRel

Theorem: (Paradigmatic effect 1.)

SpecRel ⊢ *"Relatively moving inertial observers consider different events simultaneous."*





Theorems of SpecRel

Theorem: (Paradigmatic effect 1.)

SpecRel $\vdash$ *"Relatively moving inertial observers consider different events simultaneous."*

Theorem: (Paradigmatic effect 2.)

SpecRel $\vdash$ *"Relatively moving clocks slow down."*

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Theorem: (Paradigmatic effect 3.)

SpecRel $\vdash$ *"Relatively moving spaceships shrink."*

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Theorem:

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Proof (if positive numbers have square roots).

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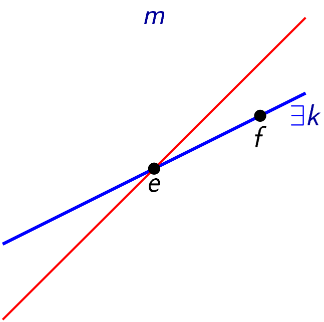
Proof (if positive numbers have square roots).
Assume indirectly that k is FTL according to m.



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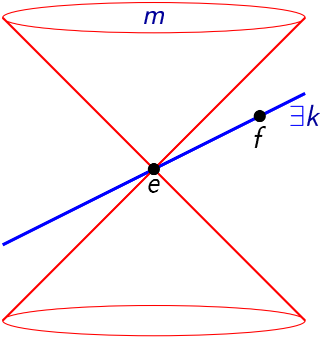

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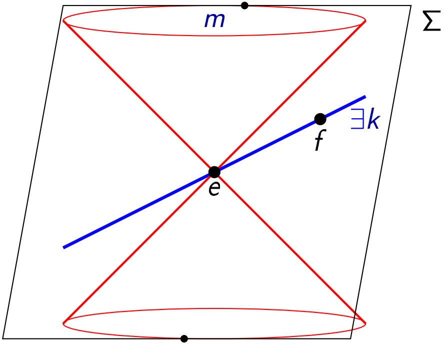

AxPh

AxOField

and $\forall x > 0\ \exists y\ x = y^2$.

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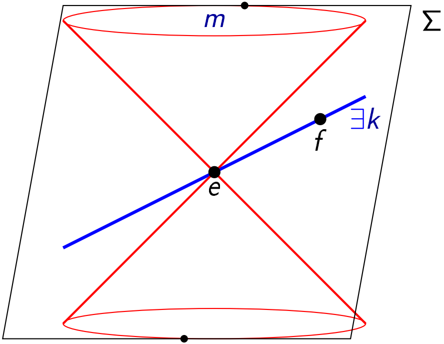

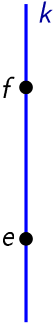

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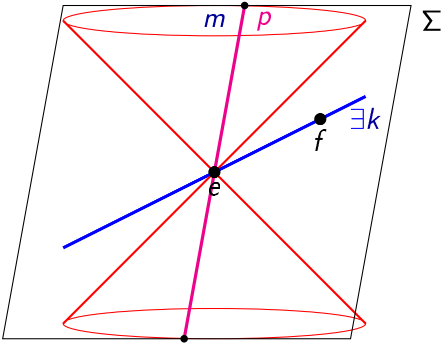

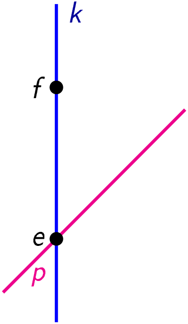

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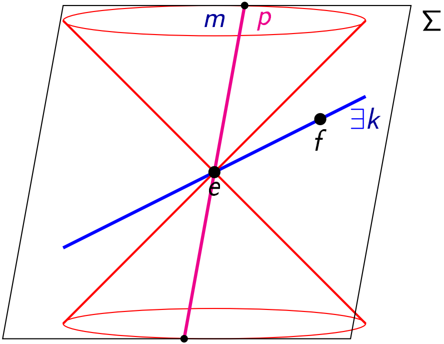

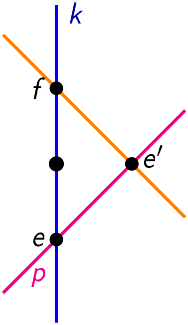

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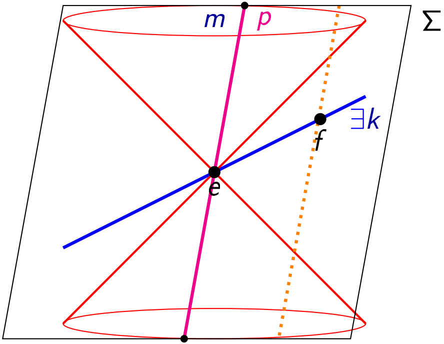

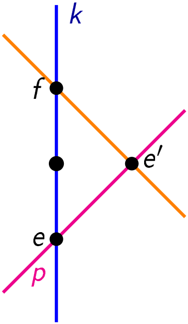

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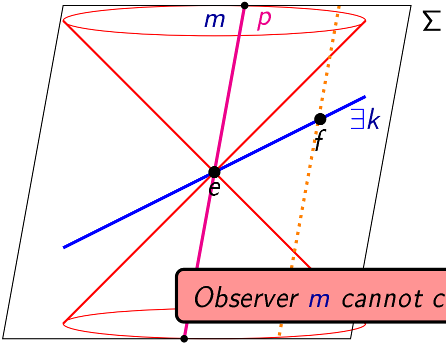

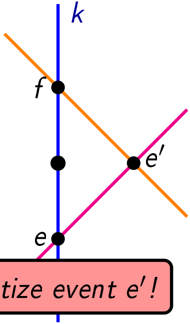

AxPh

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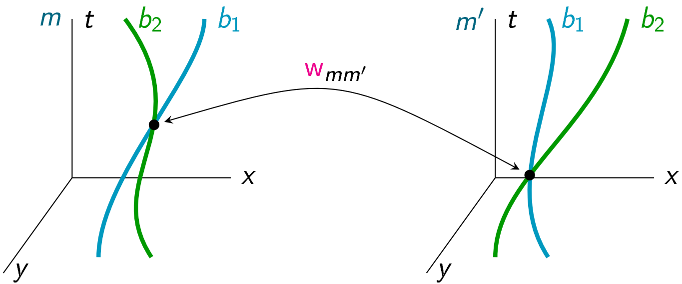
AxSelf

AxEv

Observer m cannot coordinatize event e'!

The **worldview transformation** $\mathsf{w}_{mm'}$ between observers m and m' relates the coordinate points where m and m' coordinatize the same events, i.e,:

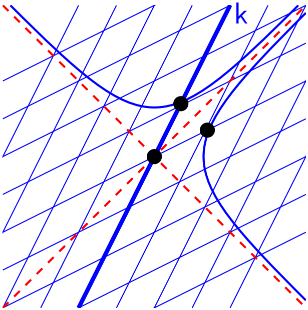
$$\mathsf{w}_{mm'}(x,y,z,t : x',y',z',t') \overset{def}{\Longleftrightarrow} \forall b\ \mathsf{W}(m,b,x,y,z,t) \iff \mathsf{W}(m',b,x',y',z',t').$$



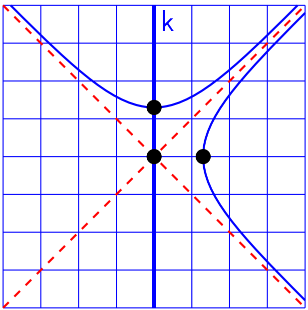
Theorems of SpecRel

Theorem:

SpecRel ⊢ *"The worldview transformations between inertial observers are Poincaré transformations."*



wordview of o



wordview of o'

SpecRel:

AxPh
AxEv
AxOField.
AxSelf
AxSym

→

Theorems:

NoFTL
Paradigmatic effects
Poincaré transf.
Etc.

What does not follow from SpecRel/SR?

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Proposition.:

$$\text{SR} \not\Rightarrow \text{Einstein's Special Relativity} \not\Rightarrow \text{SpecRel}$$

SR:=AxOField + **SPR**$^{+}$ + **AxLight** + AxEv + AxThExp + AxSelf + AxSym

SpecRel:=AxOField + **AxPh** + AxEv + AxSelf + AxSym

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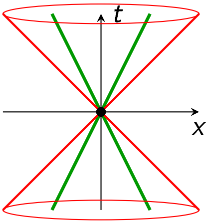
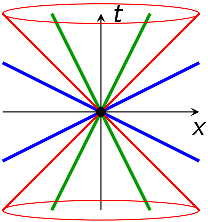
SR $\Rightarrow$ Einstein's Special Relativity $\Rightarrow$ SpecRel

Theorem:

The existence of FTL particles is logically independent of SR.





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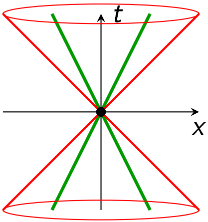

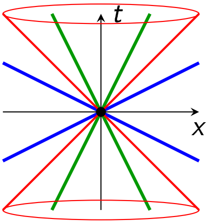

Moreover, the existence of FTL particles is also independent from relativistic particle dynamics. See Judit Madarász's talk.

Further theorems about SpecRel/SR

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SpecRel *and* SR *are consistent.*

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Theorem: (Completeness)

SpecRel and SR are complete with respect to the "standard model of SR," i.e., the Minkowski spacetimes over Euclidean ordered fields.

To be continued...