Approximating the two-variable fragment of classical predicate logic with propositional modal logics: a survey of recent results

Abstract

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It is well-known in algebraic logic that 'trouble' starts from $n \geq 3$. Just to list a few cases: Both the n-variable fragment of predicate logic, and the equational theory of its algebraic counterpart, representable cylindric algebras of dimension n (RCA_n) are undecidable for $n \geq 3$, and decidable for $n < 3$. Also, RCA_n has a finitely axiomatisable equational theory for $n < 3$, which becomes nonfinitely axiomatisable for $n \geq 3$. In our talk we survey recent results in many-dimensional modal logic, showing that from a certain perspective there can be a lot of 'trouble' in dimension 2 already.

In particular, we look at 2-dimensional cylindric set algebras as subalgebras of the full complex algebra of '2-dimensional' relational structures of the form

$$\langle U \times U, \equiv_0, \equiv_1, Id_{01} \rangle, \tag{1}$$

where, for all $\mathbf{u}, \mathbf{v} \in U \times U$,

$$\begin{aligned} \mathbf{u} \equiv_0 \mathbf{v} \quad &\iff \quad u_1 = v_1, \\ \mathbf{u} \equiv_1 \mathbf{v} \quad &\iff \quad u_0 = v_0, \\ \mathbf{u} \in Id_{01} \quad &\iff \quad u_0 = u_1. \end{aligned}$$

Instead of equations in the algebraic language having operators c_0, c_1, and d_{01}, we use formulas of the corresponding propositional multimodal language having unary diamonds $\Diamond_0$, $\Diamond_1$ (and their duals $\Box_0$, $\Box_1$) and a modal constant δ. As the variety RCA_2 is generated by cylindric set algebras, equations valid in RCA_2 correspond to multimodal formulas valid in all structures described in (1).

In this setting, the equationally expressible properties of cylindrifications and the diagonal constant in 2-dimensional representable cylindric algebras can be divided into two groups of modally expressible properties:

(i) Modal formulas saying that each $\Diamond_i$ is normal and distributes over Boolean $\vee$, and formulas expressing that $\equiv_i$ is an equivalence relation, for each $i = 0, 1$:

$$\Box_i p \to p \qquad \Box_i p \to \Box_i \Box_i p \qquad \Diamond_i p \to \Box_i \Diamond_i p. \tag{2}$$

(ii) Multimodal formulas describing 'dimension-connecting' properties of the 2-dimensional structures in (1).

A way of generalising the 2-dimensional relational structures of (1) is to consider structures where

- the universe is still a full Cartesian product of two sets, and the relations between the pairs of points still 'act coordinate-wise' (so (ii) still holds),
- the relations between the pairs of points are not necessarily equivalence relations (so (i) might not hold).

Note that this direction is kind of orthogonal to the one taken by relativised cylindric algebras [7, Section 5.5], where (i) is kept unchanged, while generalisations of (ii) are considered.

Various refinements of the properties in (2) give rise to different two-dimensional relational structures, and so to different two-dimensional modal logics. In this talk we give an overview of recent results on axiomatisation and decision problems in this area [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], describe some of the ideas behind the used methods, and give a list of open problems.

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