Strongly representable atom structures

Mohamed Khaled and Tarek Sayed Ahmed
Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

Abstract

In this article, we study various notions on atom structures. We introduce three classes for every type, and show that these are distinct if and only if the dimension is > 2. We extensively use graphs and games as introduced in algebraic logic by Hirsch and Hodkinson.

1 Introduction

In [3], Hirsch and Hodkinson proved that for finite $n \geq 3$, the class of strongly representable cylindric-type atom structures of dimension n is not definable by any set of first-order sentences: it is not elementary class. Their method depends on that RCA_n is a variety, an atomic algebra $\mathfrak{A}$ will be in RCA_n if all the equations defining RCA_n are valid in $\mathfrak{A}$. From the point of view of $At\mathfrak{A}$, each equation corresponds to a certain universal monadic second-order statement, where the universal quantifiers are restricted to ranging over the sets of atoms that are defined by elements of $\mathfrak{A}$. Such a statement will fail in $\mathfrak{A}$ if $At\mathfrak{A}$ can be partitioned into finitely many $\mathfrak{A}$-definable sets with certain properties - they call this a bad partition. This idea can be used to show that RCA_n (for $n \geq 3$) is not finitely axiomatizable, by finding a sequence of atom structures, each having some sets that form a bad partition, but with the minimal number of sets in a bad partition increasing as we go along the sequence. This can yield algebras not in RCA_n but with an ultraproduct that is in RCA_n. In this article we extend the result of Hirsch and Hodkinson to any class of strongly representable atom structure having signature between the diagonal free atom structures and the quasi polyadic equality atom structures (recall the definitions of such algebras from [1] and [2]). As in [3] we deal only with finite dimensional algebras. Fix a finite dimension $n < \omega$, with $n \geq 3$.

2 Atom structures

The action of the non-boolean operators in a completely additive atomic BAO is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra.

Definition 2.1. ***(Atom Structure)***
Let $\mathcal{A} = \langle A, +, -, 0, 1, \Omega_i : i \in I\rangle$ be an atomic boolean algebra with operators $\Omega_i : i \in I$. Let the rank of Ω_i be ρ_i. The atom structure $At\mathcal{A}$ of $\mathcal{A}$ is a relational structure

$$\langle At\mathcal{A}, R_{\Omega_i} : i \in I\rangle$$

where $At\mathcal{A}$ is the set of atoms of $\mathcal{A}$ as before, and R_{Ω_i} is a $(\rho(i)+1)$-ary relation over $At\mathcal{A}$ defined by

$$R_{\Omega_i}(a_0, \cdots, a_{\rho(i)}) \Longleftrightarrow \Omega_i(a_1, \cdots, a_{\rho(i)}) \geq a_0.$$

Similar 'dual' structure arise in other ways, too. For any not necessarily atomic BAO $\mathcal{A}$ as above, its *ultrafilter frame* is the structure

$$\mathcal{A}_+ = \langle Uf(\mathcal{A}), R_{\Omega_i} : i \in I\rangle,$$

where $Uf(\mathcal{A})$ is the set of all ultrafilters of (the boolean reduct of) $\mathcal{A}$, and for $\mu_0, \cdots, \mu_{\rho(i)} \in Uf(\mathcal{A})$, we put $R_{\Omega_i}(\mu_0, \cdots, \mu_{\rho(i)})$ iff $\{\Omega(a_1, \cdots, a_{\rho(i)}) : a_j \in \mu_j$ for $0 < j \leq \rho(i)\} \subseteq \mu_0$.

Definition 2.2. ***(Complex algebra)***
Conversely, if we are given an arbitrary structure $\mathcal{S} = \langle S, r_i : i \in I\rangle$ where r_i is a $(\rho(i)+1)$-ary relation over S, we can define its complex algebra

$$\mathfrak{Cm}(\mathcal{S}) = \langle \wp(S), \cup, \setminus, \phi, S, \Omega_i\rangle_{i\in I},$$

where $\wp(S)$ is the power set of S, and Ω_i is the $\rho(i)$-ary operator defined by

$$\Omega_i(X_1, \cdots, X_{\rho(i)}) = \{s \in S : \exists s_1 \in X_1 \cdots \exists s_{\rho(i)} \in X_{\rho(i)}, r_i(s, s_1, \cdots, s_{\rho(i)})\},$$

for each $X_1, \cdots, X_{\rho(i)} \in \wp(S)$.

It is easy to check that, up to isomorphism, $At(\mathfrak{Cm}(\mathcal{S})) \cong \mathcal{S}$ always, and $\mathcal{A} \subseteq \mathfrak{Cm}(At\mathcal{A})$ for any completely additive atomic BAO $\mathcal{A}$. If $\mathcal{A}$ is finite then of course $\mathcal{A} \cong \mathfrak{Cm}(At\mathcal{A})$.

- Atom structure of diagonal free-type algebra is $\mathcal{S} = \langle S, R_{c_i} : i < n\rangle$, where the R_{c_i} is binary relation on S.
- Atom structure of cylindric-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}} : i, j < n\rangle$, where the $R_{d_{ij}}$, R_{c_i} are unary and binary relations on S. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n\rangle$ is an atom structure of diagonal free-type.
- Atom structure of substitution-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j < n\rangle$, where the $R_{d_{ij}}$, $R_{s^i_j}$ are unary and binary relations on S, respectively. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n\rangle$ is an atom structure of diagonal free-type.
- Atom structure of quasi polyadic-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j}, R_{s_{ij}} : i, j < n\rangle$, where the R_{c_i}, $R_{s^i_j}$ and $R_{s_{ij}}$ are binary relations on S. The reducts $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n\rangle$ and $\mathfrak{Rd}_{Sc}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j < n\rangle$ are atom structures of diagonal free and substitution types, respectively.
- Atom structure of quasi polyadic equality-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}}, R_{s^i_j}, R_{s_{ij}} : i, j < n\rangle$, where the $R_{d_{ij}}$ is unary relation on S, and R_{c_i}, $R_{s^i_j}$ and $R_{s_{ij}}$ are binary relations on S.

- The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i \in I\rangle$ is an atom structure of diagonal free-type.
- The reduct $\mathfrak{Rd}_{ca}\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}} : i, j \in I\rangle$ is an atom structure of cylindric-type.
- The reduct $\mathfrak{Rd}_{Sc}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j} : i, j \in I\rangle$ is an atom structure of substitution-type.
- The reduct $\mathfrak{Rd}_{qa}\mathcal{S} = \langle S, R_{c_i}, R_{s^i_j}, R_{s_{ij}} : i, j \in I\rangle$ is an atom structure of quasi polyadic-type.

Definition 2.3. *An algebra is said to be representable if and only if it is isomorphic to a subalgebra of a direct product of set algebras of the same type.*

Definition 2.4. *Let $\mathcal{S}$ be an n-dimensional algebra atom structure. $\mathcal{S}$ is strongly representable if every atomic n-dimensional algebra $\mathcal{A}$ with $At\mathcal{A} = \mathcal{S}$ is representable. We write $SDfS_n$, SCS_n, $SSCS_n$, SQS_n and $SQES_n$ for the classes of strongly representable (n-dimensional) diagonal free, cylindric, substitution, quasi polyadic and quasi polyadic equality algebra atom structures, respectively.*

Note that for any n-dimensional algebra $\mathcal{A}$ and atom structure $\mathcal{S}$, if $At\mathcal{A} = \mathcal{S}$ then $\mathcal{A}$ embeds into $\mathfrak{Cm}\mathcal{S}$, and hence $\mathcal{S}$ is strongly representable iff $\mathfrak{Cm}\mathcal{S}$ is representable.

3 Graphs and Strong representability

In this section, by a graph we will mean a pair $\Gamma = (G, E)$, where $G \neq \phi$ and $E \subseteq G \times G$ is a reflexive and symmetric binary relation on G. We will often use the same notation for Γ and for its set of nodes (G above). A pair $(x, y) \in E$ will be called an edge of Γ. See [5] for basic information (and a lot more) about graphs.

Definition 3.1. *Let $\Gamma = (G, E)$ be a graph.*

1. *A set $X \subset G$ is said to be independent if $E \cap (X \times X) = \phi$.*
2. *The chromatic number $\chi(\Gamma)$ of Γ is the smallest $\kappa < \omega$ such that G can be partitioned into κ independent sets, and ∞ if there is no such κ.*

Definition 3.2.

- *For an equivalence relation $\sim$ on a set X, and $Y \subseteq X$, we write $\sim\restriction Y$ for $\sim \cap (Y \times Y)$. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i)$, $K(j)$ are both undefined, or they are both defined and are equal.*
- *For any two relations $\sim$ and $\approx$. The composition of $\sim$ and $\approx$ is the set*

$$\sim \circ \approx = \{(a, b) : \exists c(a \sim c \wedge c \approx b)\}.$$

Definition 3.3. *Let Γ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n\rangle$ as follows:*

1. *H is the set of all pairs $(K, \sim)$ where $K : n \to \Gamma \times n$ is a partial map and $\sim$ is an equivalent relation on n satisfying the following conditions*

   (a) *If $|n/\sim| = n$, then $dom(K) = n$ and $rng(K)$ is not independent subset of n.*

   (b) *If $|n/\sim| = n - 1$, then K is defined only on the unique $\sim$ class $\{i, j\}$ say of size 2 and $K(i) = K(j)$.*

   (c) *If $|n/\sim| \leq n - 2$, then K is nowhere defined.*

2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.

3. $(K, \sim) \equiv_i (K', \sim')$ *iff* $K(i) = K'(i)$ *and* $\sim\upharpoonright (n \setminus \{i\}) = \sim' \upharpoonright (n \setminus \{i\})$.

4. $(K, \sim) \equiv_{ij} (K', \sim')$ *iff* $K(i) = K'(j)$, $K(j) = K'(i)$, *and* $K(\kappa) = K'(\kappa) (\forall \kappa \in n \setminus \{i, j\})$ *and if* $i \sim j$ *then* $\sim = \sim'$, *if not, then* $\sim' = \sim \circ [i, j]$.

It may help to think of $K(i)$ as assigning the nodes $K(i)$ of $\Gamma \times n$ not to i but to the set $n \setminus \{i\}$, so long as its elements are pairwise non-equivalent via $\sim$. For a set X, $\mathcal{B}(X)$ denotes the boolean algebra $\langle \wp(X), \cup, \setminus \rangle$. We write $a \cap b$ for $-(-a \cup -b)$.

Definition 3.4. *Let $\mathfrak{B}(\Gamma) = \langle \mathcal{B}(\eta(\Gamma)), c_i, s^i_j, s_{ij}, d_{ij} \rangle_{i,j<n}$ be the algebra, with extra non-Boolean operations defined as follows:*

$$\begin{aligned} d_{ij} &= D_{ij}, \\ c_i X &= \{c : \exists a \in X, a \equiv_i c\}, \\ s_{ij} X &= \{c : \exists a \in X, a \equiv_{ij} c\}, \\ s^i_j X &= \begin{cases} c_i(X \cap D_{ij}), & \text{if } i \neq j, \\ X, & \text{if } i = j. \end{cases} \quad \text{For all } X \subseteq \eta(\Gamma). \end{aligned}$$

Definition 3.5. *For any $\tau \in \{\pi \in n^n : \pi$ is a bijection$\}$, and any $(K, \sim) \in \eta(\Gamma)$. We define $\tau(K, \sim) = (K \circ \tau, \sim \circ \tau)$.*

The proof of the following two Lemmas is straightforward.

Lemma 3.1.
For any $\tau \in \{\pi \in n^n : \pi$ is a bijection$\}$, and any $(K, \sim) \in \eta(\Gamma)$. $\tau(K, \sim) \in \eta(\Gamma)$.

Lemma 3.2.
For any $(K, \sim)$, $(K', \sim')$, and $(K'', \sim'') \in \eta(\Gamma)$, and $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim')$.

2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim')$.

3. *If $(K, \sim) \equiv_{ij} (K', \sim')$, and $(K, \sim) \equiv_{ij} (K'', \sim'')$, then $(K', \sim') = (K'', \sim'')$.*

4. *If $(K, \sim) \in D_{ij}$, then*
   $(K, \sim) \equiv_i (K', \sim') \iff \exists (K_1, \sim_1) \in \eta(\Gamma) : (K, \sim) \equiv_j (K_1, \sim_1) \wedge (K', \sim') \equiv_{ij} (K_1, \sim_1)$.

5. $s_{ij}(\eta(\Gamma)) = \eta(\Gamma)$.

Theorem 3.1. *For any graph Γ, $\mathfrak{B}(\Gamma)$ is a simple QEA_n.*

Proof. We follow the axiomatization in [2] except renaming the items by Q_i. Let $X \subseteq \eta(\Gamma)$, and $i, j, \kappa \in n$:

- $s_i^i = ID$ by definition 3.4, $s_{ii}X = \{c : \exists a \in X, a \equiv_{ii} c\} = \{c : \exists a \in X, a = c\} = X$ (by Lemma 3.2 (1));
  $s_{ij}X = \{c : \exists a \in X, a \equiv_{ij} c\} = \{c : \exists a \in X, a \equiv_{ji} c\} = s_{ji}X$ (by Lemma 3.2 (2)).
- Axioms Q_1, Q_2 follow directly from the fact that the reduct $\mathfrak{Rd}_{ca}\mathfrak{B}(\Gamma) = \langle \mathcal{B}(\eta(\Gamma)), c_i, d_{ij}\rangle_{i,j<n}$ is a cylindric algebra which is proved in [3].
- Axioms Q_3, Q_4, Q_5 follow from the fact that the reduct $\mathfrak{Rd}_{ca}\mathfrak{B}(\Gamma)$ is a cylindric algebra, and from [1] (Theorem 1.5.8(i), Theorem 1.5.9(ii), Theorem 1.5.8(ii)).
- s_j^i is a boolean endomorphism by [1] (Theorem 1.5.3).

$$\begin{aligned} s_{ij}(X \cup Y) = \quad & \{c : \exists a \in (X \cup Y), a \equiv_{ij} c\} \\ = \quad & \{c : (\exists a \in X \vee \exists a \in Y), a \equiv_{ij} c\} \\ = \quad & \{c : \exists a \in X, a \equiv_{ij} c\} \cup \{c : \exists a \in Y, a \equiv_{ij} c\} \\ = \quad & s_{ij}X \cup s_{ij}Y. \end{aligned}$$

  $s_{ij}(-X) = \{c : \exists a \in (-X), a \equiv_{ij} c\}$, and $s_{ij}X = \{c : \exists a \in X, a \equiv_{ij} c\}$ are disjoint. For, let $c \in (s_{ij}(X) \cap s_{ij}(-X))$, then $\exists a \in X \wedge b \in (-X)$, such that $a \equiv_{ij} c$, and $b \equiv_{ij} c$. Then $a = b$, (by Lemma 3.2 (3)), which is a contradiction. Also,

$$\begin{aligned} s_{ij}X \cup s_{ij}(-X) = \quad & \{c : \exists a \in X, a \equiv_{ij} c\} \cup \{c : \exists a \in (-X), a \equiv_{ij} c\} \\ = \quad & \{c : \exists a \in (X \cup -X), a \equiv_{ij} c\} \\ = \quad & s_{ij}\eta(\Gamma) \\ = \quad & \eta(\Gamma). \text{ (by Lemma 3.2 (5))} \end{aligned}$$

  therefore, s_{ij} is a boolean endomorphism.

- $$\begin{aligned} s_{ij}s_{ij}X = \quad & s_{ij}\{c : \exists a \in X, a \equiv_{ij} c\} \\ = \quad & \{b : (\exists a \in X \wedge c \in \eta(\Gamma)), a \equiv_{ij} c, \text{ and } c \equiv_{ij} b\} \\ = \quad & \{b : \exists a \in X, a = b\} \\ = \quad & X. \end{aligned}$$

- $$\begin{aligned} s_{ij}s_j^iX = \quad & \{c : \exists a \in s_j^iX, a \equiv_{ij} c\} \\ = \quad & \{c : \exists b \in (X \cap d_{ij}), a \equiv_i b \wedge a \equiv_{ij} c\} \\ = \quad & \{c : \exists b \in (X \cap d_{ij}), c \equiv_j b\} \text{ (by Lemma 3.2 (4))} \\ = \quad & s_i^jX. \end{aligned}$$

- We need to prove that $s_{ij}s_{i\kappa}X = s_{j\kappa}s_{ij}X$ if $|\{i,j,\kappa\}| = 3$. For, let $(K,\sim) \in s_{ij}s_{i\kappa}X$ then $\exists (K',\sim') \in \eta(\Gamma)$, and $\exists (K'',\sim'') \in X$ such that $(K'',\sim'') \equiv_{i\kappa} (K',\sim')$ and $(K',\sim') \equiv_{ij} (K,\sim)$.
  Define $\tau : n \to n$ as follows:

$$\begin{aligned} \tau(i) = &\quad j \\ \tau(j) = &\quad \kappa \\ \tau(\kappa) = &\quad i, \text{ and} \\ \tau(l) = &\quad l \text{ for every } l \in (n \setminus \{i,j,\kappa\}). \end{aligned}$$

  Now, it is easy to verify that $\tau(K',\sim') \equiv_{ij} (K'',\sim'')$, and $\tau(K',\sim') \equiv_{j\kappa} (K,\sim)$. Therefore, $(K,\sim) \in s_{j\kappa}s_{ij}X$, i.e., $s_{ij}s_{i\kappa}X \subseteq s_{j\kappa}s_{ij}X$. Similarly, we can show that $s_{j\kappa}s_{ij}X \subseteq s_{ij}s_{i\kappa}X$.

- Axiom Q_{10} follows from [1] (Theorem 1.5.7)

- Axiom Q_{11} follows from axiom 2, and the definition of s^i_j.

Since $\mathfrak{Rd}_{ca}\mathfrak{B}$ is a simple CA_n, by [3], then $\mathfrak{B}$ is simple. □

Definition 3.6. *Let $\mathfrak{C}(\Gamma)$ be the subalgebra of $\mathfrak{B}(\Gamma)$ generated by the set of atoms.*

Note that the cylindric algebra constructed in [3] is $\mathfrak{Rd}_{ca}\mathfrak{B}(\Gamma)$ not $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$, but all results in [3] can be applied to $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$. Therefore, since our results depends basically on [3], we will refer to [3] directly when we apply it to catch any result about $\mathfrak{Rd}_{ca}\mathfrak{C}(\Gamma)$.

Theorem 3.2. *$\mathfrak{C}(\Gamma)$ is a simple QEA_n generated by the set of the $n-1$ dimensional elements.*

Proof. $\mathfrak{C}(\Gamma)$ is a simple QEA_n from Theorem 3.1. It remains to show that $\{(K,\sim)\} = \prod\{c_i\{(K,\sim)\} : i < n\}$ for any $(K,\sim) \in H$. Let $(K,\sim) \in H$, clearly $\{(K,\sim)\} \leq \prod\{c_i\{(K,\sim)\} : i < n\}$. For the other direction assume that $(K',\sim') \in H$ and $(K,\sim) \neq (K',\sim')$. We show that $(K',\sim') \notin \prod\{c_i\{(K,\sim)\} : i < n\}$. Assume toward a contradiction that $(K',\sim') \in \prod\{c_i\{(K,\sim)\} : i < n\}$, then $(K',\sim') \in c_i\{(K,\sim)\}$ for all $i < n$, i.e., $K'(i) = K(i)$ and $\sim' \restriction (n \setminus \{i\}) = \sim \restriction (n \setminus \{i\})$ for all $i < n$. Therefore, $(K,\sim) = (K',\sim')$ which makes a contradiction, and hence we get the other direction. □

Theorem 3.3. *Let Γ be a graph.*

1. *Suppose that $\chi(\Gamma) = \infty$. Then $\mathfrak{C}(\Gamma)$ is representable.*

2. *If Γ is infinite and $\chi(\Gamma) < \infty$ then $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable.*

Proof.

1. We have $\mathfrak{Rd}_{ca}\mathfrak{C}$ is representable (c.f., [3]). Let $X = \{x \in \mathfrak{C} : \Delta x \neq n\}$. Call $J \subseteq \mathfrak{C}$ inductive if $X \subseteq J$ and J is closed under infinite unions and complementation. Then $\mathfrak{C}$ is the smallest inductive subset of $\mathfrak{C}$. Let f be an isomorphism of $\mathfrak{Rd}_{ca}\mathfrak{C}$ onto a cylindric set algebra with base U. Clearly, by definition, f preserves s^i_j for each $i,j < n$. It remains to show that f preserves s_{ij} for every $i,j < n$. Let $i,j < n$, since s_{ij} is

boolean endomorphism and completely additive, it suffices to show that $fs_{ij}x = s_{ij}fx$ for all $x \in At\mathfrak{C}$. Let $x \in At\mathfrak{C}$ and $\mu \in n \setminus \Delta x$. If $\kappa = \mu$ or $l = \mu$, say $\kappa = \mu$, then

$$fs_{\kappa l}x = fs_{\kappa l}c_\kappa x = fs_l^\kappa x = s_l^\kappa fx = s_{\kappa l}fx.$$

If $\mu \notin \{\kappa, l\}$ then

$$fs_{\kappa l}x = fs_\mu^l s_l^\kappa s_\kappa^\mu c_\mu x = s_\mu^l s_l^\kappa s_\kappa^\mu c_\mu fx = s_{\kappa l}fx.$$

2. Assume toward a contradiction that $\mathfrak{Rd}_{df}\mathfrak{C}$ is representable. Since $\mathfrak{Rd}_{ca}\mathfrak{C}$ is generated by $n-1$ dimensional elements then $\mathfrak{Rd}_{ca}\mathfrak{C}$ is representable. But this contradicts Proposition 5.4 in [3].

□

Theorem 3.4.
Let $2 < n < \omega$ and $\mathcal{T}$ be any signature between Df_n and QEA_n. Then the class of strongly representable atom structures of type $\mathcal{T}$ is not elementary.

Proof. By Erdös's famous 1959 Theorem [4], for each finite κ there is a finite graph G_κ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let Γ_κ be the disjoint union of the G_l for $l > \kappa$. Clearly, $\chi(\Gamma_\kappa) = \infty$. So by Theorem 3.3 (1), $\mathfrak{C}(\Gamma_\kappa) = \mathfrak{C}(\Gamma_\kappa)^+$ is representable.
Now let Γ be a non-principal ultraproduct $\prod_D \Gamma_\kappa$ for the Γ_κ. It is certainly infinite. For $\kappa < \omega$, let σ_κ be a first-order sentence of the signature of the graphs. stating that there are no cycles of length less than κ. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Łoś's Theorem, $\Gamma \models \sigma_\kappa$ for all κ. So Γ has no cycles, and hence by, [3] Lemma 3.2, $\chi(\Gamma) \leq 2$. By Theorem 3.3 (2), $\mathfrak{Rd}_{df}\mathfrak{C}$ is not representable. It is easy to show (e.g., because $\mathfrak{C}(\Gamma)$ is first-order interpretable in Γ, for any Γ) that

$$\prod_D \mathfrak{C}(\Gamma_\kappa) \cong \mathfrak{C}(\prod_D \Gamma_\kappa).$$

Combining this with the fact that: for any n-dimensional atom structure $\mathcal{S}$

$$\mathcal{S} \text{ is strongly representable} \iff \mathfrak{Cm}\mathcal{S} \text{ is representable,}$$

the desired follows. □

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