Classification of absorbent-continuous, sharp FL_e-algebras on weakly real chains

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Abstract

FL_e-algebras are algebraic models of the substructural logic $\mathsf{FL_e}$. The classification of absorbent-continuous, sharp FL_e-algebras over weakly real chains is given: The algebra is determined by its negative cone, and the related cone operation can only be chosen from a certain subclass of BL-algebras. It is shown that absorbent-continuity is the most relaxed version of the naturally ordered condition under which the classification theorem holds. The classification theorem does not hold either if the algebra is not sharp.

Keywords: Substructural logics, Residuated lattice, involutive FL_e-algebra, ordinal sum, twin-rotation, classification

1 Introduction

Residuated lattices have been introduced in the 30s of the last century by Ward and Dilworth [25] to investigate ideal theory of commutative rings with unit. Examples of residuated lattices include Boolean algebras, Heyting algebras [18], MV-algebras [3], BL-algebras, [7] and lattice-ordered groups; a variety of other algebraic structures can be rendered as residuated lattices. The topic did not become a leading trend on its own right back then. Nowadays the investigation of residuated lattices (roughly, residuated monoids on lattices) has got a new impetus and has been staying in the focus of strong

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international attention. Beyond the algebraic interest, the reason is that residuated lattices turned out to be algebraic counterparts of substructural logics [24, 23]. Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic and their non-commutative versions. These logics had different motivations and methodology. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella. Residuated lattices themselves have been the key component in this remarkable unification. An extensive monograph about residuated lattices and substructural logics went to print in 2007 [6]. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science. FL_e-algebras are commutative residuated lattices with an additional constant. For FL_e-algebras, those with an involutive negation are of special interest. Involutive FL_e-algebras have very interesting symmetry properties [11, 12, 10, 19] and, as a consequence, for involutive FL_e-algebras we have beautiful geometric constructions which are lacking for general FL_e-algebras [11, 17, 20]. Furthermore, not only involutive FL_e-algebras have very interesting symmetry properties, but some of their logical calculi have important symmetry properties too: Both sides of a sequent may contain more than one formula, while (hyper)sequent calculi for their non-involutive counterparts admit at most one formula to the right.

As for the *classification problem* of residuated lattices, as one naturally expects, it is possible only by imposing additional postulates. A first precursor is due to Hölder who proved in [8] that every *cancellative*, *Archimedean*, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers. Aczél used tools of analysis to investigate continuous semigroup operations over intervals of real numbers[1] and also found in [1, page 256] the cancellative property[2] to be sufficient and necessary for the existence of an order-isomorphism to a subsemigroup of the additive semigroup of the real numbers [1, page 268]. Clifford showed in [4] that every *Archimedean*, naturally and totally ordered semigroup in which the *cancellation law does not hold* can be embedded into either the real numbers in the interval $[0,1]$ with the usual ordering and $ab = \max(a+b, 1)$ or the real numbers in the interval $[0,1]$ and the symbol ∞ with the usual ordering and $ab = a + b$ if $a + b \leq 1$ and $ab = \infty$ if $a + b > 1$. For a summary of the Hölder and Clifford theorems, see [5, Theorem 2 in Section

[1] Isotonicity of the semigroup operation is not assumed.

[2] He called is reducible.

2 of Chapter XI]. Clifford also introduced the ordinal sum construction for a family of totally ordered semigroups in [4] and proved that every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinial sum of a totally ordered set of ordinally irreducible such semigroups. Mostert and Shields gave a complete description of topological semigroups over compact manifolds with connected, regular boundary in [22] by using a subclass of compact connected Lie groups and via classifying semigroups on arcs such that one endpoint functions as an identity for the semigroup, and the other functions as a zero. They classified such semigroups as ordinal sums of three basic multiplications which an arc may possess. The word 'topological' refers to the continuity of the semigroup operation with respect to the topology. In the next related classification result, the topologically connected property of the underlying chain was dropped whereas the continuity condition was somewhat strengthened: Under the assumption of divisibility[3] [4] [5], residuated chains were classified as ordinal sums[6] of linearly ordered Wajsberg hoops in [2]. Postulating the divisibility condition proved to be sufficient for the classification of residuated monoids over arbitrary lattices, see [9], where the authors introduced the notion of poset sum of hoops, a common generalization of ordinal sum and of direct product. They proved that every commutative and divisible residuated lattice embeds into the poset sum of a family of MV-chains and that the embedding is an isomorphism in the finite case. Next, SIU-algebras over arbitrary lattices were classified in [16], see Theorem 3 below. Here the authors assume the existence of a dual-isomorphism between the positive and negative cones of the algebra. For SIU-algebras over weakly real chains, this condition is equivalent to postulating divisibility only for the negative cone of the algebra. In the present paper we classify a class of residuated lattices by assuming only a very weak form of continuity, called absorbent-continuity. It is a much

[3] Divisibility is the dual notion of the naturally ordered property; here semigroups are negatively ordered.

[4] For residuated integral monoids, divisibility is equivalent to the continuity of the semigroup operation in the order topology if the underlying chain is order dense.

[5] Divisibility is the algebraic analogue of the Intermediate Value Theorem in real analysis, and for residuated integral monoids over order-dense chains, it can be considered a stronger version of continuity of the monoidal operation than the continuity of it with respect to the order topology. Indeed, divisibility entails continuity on order-dense chains as mentioned in the previous footnote. On the other hand, if the order topology of the chain is the discrete one then every operation is continuous but obviously not all operations obey the divisibility condition.

[6] The notion of ordinal sum has slightly been modified to ease the formulation of this result.

weaker condition than even the continuity condition of SIU-algebras; in fact, it is the weakest possible continuity condition under which the statement of our classification theorem still holds.

The goal of this paper is to classify absorbent-continuous, sharp FL_e-algebras on weakly real chains. Surprisingly, the restriction of those monoids to their negative cone is necessarily continuous (everywhere) in the order topology of their underlying chain. Equivalently, one may say that the restriction of those monoids to their negative cones is necessarily divisible, as divisibility and continuity are equivalent in our setting. The result holds only under the sharpness condition, and hence a classification for involutive FL_e-monoids is still lacking, but in any case the result is very surprising, as involutive *integral* monoids over chains, that can be more specific than weakly real chains, may have discontinuities even below the fixed point of their negation.

While for involutive integral monoids and even for involutive t-norms a classification is still lacking, for sharp FL_e-algebras on weakly real chains we obtain here a classification. Since $[0, 1]$, the unit interval of real numbers, is a weakly real chain, our result also provides with the classification of absorbent-continuous, sharp uninorms. Finally, we show that absorbent-continuity can not be omitted from the conditions of the classification. Also we show that closed intervals of the real numbers are not the only example weakly real chains.

2 Preliminaries

Definition 1 We call a chain $\langle X, \leq \rangle$ *weakly real* if

1. X is order-dense and complete,
2. there exists a dense $Y \subset X$ with $|Y| < |X|$, and
3. for any $x, y \in Y$ there exist $u, v \in Y$ such that $u > x$, $v > y$, and there exists a strictly increasing function from $[x, u]$ into $[y, v]$.

Definition 2 A commutative binary operation $\circledast$ on a poset $(X, \leq)$ is called *residuated* if there exists another binary operation $\rightarrow_{\circledast}$ on X such that for x, y, z, $x \circledast y \leq z$ iff $y \rightarrow_{\circledast} z \geq x$. Call $\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ an *FL_e-monoid* if $\mathcal{C} = \langle X, \leq \rangle$ is a poset, $(X, \circledast)$ is a commutative, residuated monoid over $\mathcal{C}$ with neutral element t, and f is an arbitrary constant. If X is a lattice, we speak about *FL_e-algebras*. Define the positive and the negative *cone* of $\mathcal{U}$ by $X^+ = \{x \in X \mid x \geq t\}$ and $X^- = \{x \in X \mid x \leq t\}$, respectively. Both cones are closed with respect to the monoidal operation $\circledast$; throughout the paper we will denote the negative and the positive cone operation of $\circledast$, by $\otimes$ and $\oplus$, respectively. Call $\mathcal{U}$ *conic* if every element of X is comparable with t, that is, if $X = X^+ \cup X^-$. Call $\mathcal{U}$ *representable* if it can be represented as subdirect product of chains. Call $\mathcal{U}$ *finite* if X is a finite set, *bounded* if X has top $\top$ and bottom element $\bot$. If X is linearly ordered, we speak about FL_e*-chains*. Call $\mathcal{U}$ *involutive*, if for $x \in X$, $(x')' = x$ holds, where $x' = x \to_\circledast f$. Call an involutive FL_e-monoid *sharp*, if $t = f$. Call a representable, bounded, sharp FL_e-monoid a *SIU-algebra*, if for $x, y \in X^-$, $x' \circledast y' = (x \circledast y)'$ holds. A po-monoid is *integral* (resp. *dually integral*) if it has a top (resp. bottom) element which is also the unit element of $\circledast$. Monoidal operations of FL_e-algebras over $[0, 1]$ are called *uninorms*, of integral FL_e-algebras over $[0, 1]$ are called *t-norms*. *BL-algebras* are divisible, representable, bounded, integral FL_e-algebras with $f = \bot$. *MV-algebras* are BL-algebras satisfying $x'' = x$. *Hoops* are divisible, commutative integral residuated po-monoids. *Wajsberg* hoops are MV-algebras deprived of $\bot$. *Commutative residuated lattices* are exactly the f-free reducts of FL_e-algebras.

Any residuated operation it is also partially ordered (isotone), and therefore, $' : X \to X$ is an order-reversing involution. A residuated operation on an order-dense chain (viewed as a two-place function) is left-continuous in the square of the order topology.

We recall a result from [4]. This theorem discusses a certain way of constructing a new semigroup from a family of semigroups.

Definition 3 (Ordinal sum construction - Clifford sense) *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta}$ is both the unit element of G_α and the annihilator of G_β, and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases} \tag{1}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative. We call G the ordinal sum of the G_α's, and each G_α will be called a summand of G.

Definition 4 (Ordinals sum construction - Aglianó-Montagna sense) [2]

Let $(I, \leq)$ be a totally ordered set. For each $i \in I$ let $\mathbf{A}_i = \langle A_i, \cdot_i, \rightarrow_i, 1\rangle$ be a hoop such that for every $i \neq i$, $A_i \cap A_j = 1$. Then we can define the ordinal sum as the hoop $\bigoplus_{i\in I} \mathbf{A}_i = \langle \cup_{i\in I}, \cdot, \rightarrow, 1\rangle$ where the operations $\cdot, \rightarrow_i$ are given by:

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if} \quad x, y \in A_i, \\ x & \text{if} \quad x \in A_i \setminus \{1\}, y \in A_j \quad \text{and} \quad i < j, \\ y & \text{if} \quad y \in A_i \setminus \{1\}, x \in A_j \quad \text{and} \quad i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} 1 & \text{if} \quad x \in A_i \setminus \{1\}, y \in A_j \quad \text{and} \quad i < j, \\ x \rightarrow_i y & \text{if} \quad x, y \in A_i, \\ y & \text{if} \quad y \in A_i, x \in A_j \quad \text{and} \quad i < j. \end{cases}$$

If in addition I has a minimum i_0 and $\mathbf{A}_{i_0}$ is a bounded hoop, then $\bigoplus_{i\in I} \mathbf{A}_i$ denotes the bounded hoop whose operations $\cdot, \rightarrow_i$ are defined as before, and whose bottom element is the minimum of $\mathbf{A}_{i_0}$. Each $\mathbf{A}_i$ is called a component of $\bigoplus_{i\in I} \mathbf{A}_i$.

In order to ease the distinction between the two ordinal sum constructions, we will speak about summands in case of Clifford-style ordinal sums, whereas we will speak about components in case of Aglianó-Montagna-style ordinal sums.

Theorem 1 [2] *Every totally ordered BL-algebra is the ordinal sum of a family of Wajsberg hoops, whose first component is an MV-algebra.*

The twin-rotation construction was introduced in [17]. Here we need a special case of it:

Definition 5 (Twin-rotation construction – sharp case) Let X_1 be a partially ordered set with top element t, and and X_2 be a partially ordered set with bottom element t such that the connected ordinal sum $os_c\langle X_1, X_2\rangle$ of X_1 and X_2 (that is putting X_1 under X_2, and identifying the top of X_1 with the bottom of X_2) has an order reversing involution $'$ with fixed point t. Denote the partial order of $os_c\langle X_1, X_2\rangle$ also by $\leq$. Let $(X_1, \otimes)$ and $(X_2, \oplus)$ be commutative semigroups, both with neutral element t. Assume that $(X_1, \otimes)$ is residuated and assume that all residua $x \rightarrow_\oplus y$

exist if $x, y \in X_2$, $x \leq y$.[7] Denote

$$\mathcal{U}_{\otimes}^{\oplus} = \langle os_c\langle X_1, X_2\rangle, \circledast, \leq, t, t\rangle$$

where $\circledast$ is defined as follows:

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_{\oplus} y')' & \text{if } x \in X_2,\, y \in X_1, \text{ and } x \leq y' \\ (y \to_{\oplus} x')' & \text{if } x \in X_1,\, y \in X_2, \text{ and } x \leq y' \\ (y \to_{\otimes} x')' & \text{if } x \in X_2,\, y \in X_1, \text{ and } x \not\leq y' \\ (x \to_{\otimes} y')' & \text{if } x \in X_1,\, y \in X_2, \text{ and } x \not\leq y' \end{cases} . \qquad (2)$$

Call $\circledast$ (resp. $\mathcal{U}_{\otimes}^{\oplus}$) the twin-rotation of $\otimes$ and $\oplus$ (resp. of the first and the second partially ordered monoid).

Theorem 2 [17] **(Conic Representation Theorem – sharp case)** *Any conic, sharp FL_e-monoid can be represented as the twin-rotation of its negative and positive cone.*

We will also rely on the classification of SIU-algebras:

Theorem 3 [16] *$\mathcal{U} = \langle X, \circledast, \leq, t, f\rangle$ is a SIU-algebra if and only if its negative cone is a BL-algebra with components which are either cancellative or MV-algebras with two elements, and with no two consecutive cancellative component, $\oplus$ is the dual of $\otimes$ with respect to $'$, and $\circledast$ is given by (2).*

One of the two main tools in proving our classification theorem in Theorem 5 is the result in Lemma 4 about the relationship of two operations, both of which are derived from $\circledast$, as follows:

Definition 6 For a commutative complete residuated chain $\langle X, \leq, \circledast, \to, 1\rangle$ and for $x, y \in X \setminus \{\top\}$ define

$$\begin{aligned} x \circledast_{co}\, y &= \inf\{x_1 \circledast y_1 \mid x_1 > x, y_1 > y\}, \\ x \circledast_{Q}\, y &= \inf\{x \circledast y_1 \mid y_1 > y\}. \end{aligned}$$

Call $\circledast_{co}$ and $\circledast_{Q}$ the *skewed modification* [12, 10] and the *companion* [19] of $\circledast$, respectively.

[7] This means that for $x, y \in X_2$ and $x \leq y$, the maximal element of the set $\{z \mid x \oplus z \leq y\}$ exists.

In addition, assume X is a *order-dense*. Then $x \circledast_{co} y = x \circledast y$ iff (x, y) is a continuity point of $\circledast$ (viewed as a two-place function) in the order topology of the chain. Then, for $\circledast$ being residuated is known to be equivalent to being left-continuous, as a two-place function (in the order topology) whereas being co-residuated is known to be equivalent to being right-continuous. By using that the chain is order-dense to-gether with that of the monotonicity of $\circledast$, it is an easy exercise to prove that $x \circledast_{co} y$ is equal to the limit of $x_i \circledast y_i$, x_i and y_i are being arbitrarily chosen sequences with $x_i > x$ and $y_i > y$, converging to x and y, respectively. The skewed modification is right-continuous, by definition, therefore it is always a co-residuated operation since the chain is complete, that is, it is residuated with respect to $\geq$, the dual ordering relation.

3 Classification

In [12, Corollary 4] (see as well [10]) it has been demonstrated that the skewed modification and the companion of $\circledast$ *coincide* whenever $\circledast$ is an integral FL_e-algebra on $[0, 1]$, such that $x \mapsto x \rightarrow_{\circledast} 0$ is an involution of $[0, 1]$ with fixed point a, and $x \mapsto x \rightarrow_{\circledast} a$, is an involution of $[a, 1]$. We will prove in this paper that $\circledast_{co}$ and $\circledast_Q$ also coincide when $\circledast$ is any sharp FL_e-algebra on a weakly real chain.

Lemma 4 *Let $\langle X, \circledast, \leq, t, f \rangle$ be a sharp FL_e-algebra on a weakly real chain. For $\top \neq x, y \in X$,*

$$x \circledast_{co} y = x \circledast_Q y$$

holds. ■

In this section we present the main theorem of our paper (Theorem 5), which states that $\mathcal{U}$ is an absorbent-continuous, sharp FL_e-algebra on a weakly real chain if and only if the negative cone of $\mathcal{U}$ is a BL-chain with components which are either cancellative (that is, those components are negative cones of totally ordered Abelian groups) or MV-algebras with two elements, and with no two consecutive cancellative components, $\oplus$ is the dual of $\otimes$ with respect to $'$, and $\circledast$ is given by (2). In other words, each absorbent-continuous, sharp FL_e-algebra on a weakly real chain is a SIU-chain. This theorem can also be read as follows: For sharp FL_e-algebras on weakly real chains it is sufficient (and as we will see in Example 6, also necessary) to assume absorbent-continuity, which is a very relaxed version of the naturally ordered condition on the negative cone operation, and it implies continuity on the whole negative cone. To this end, we first introduce and investigate the *absorbent function* of the monoidal operation of involutive FL_e-algebras. Introduced in [14] under the name 'skeleton', the absorbent region of a left-continuous t-norm is the subset of its domain, where its value equals with the minimum of the arguments. Exploitation of this notion leads to the second main tool in proving our classification theorem.

Definition 7 For an involutive FL_e-monoid $\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ on a complete poset let

$$A(x) = \begin{cases} \max\{u \in X^+ \mid u \oplus x = x\}, & \text{if } x \in X^+ \\ (\inf\{u \in X^- \mid u \otimes x = x\})', & \text{if } x \in X^- \end{cases}$$

and call it the *absorbent function* of $\circledast$. The maximum of the set in the first line always exists since $\oplus$ is residuated and the infimum in the second line exists since the poset is complete.

Definition 8 Let $\langle X, \circledast, \leq, t, f \rangle$ be a sharp FL_e-monoid on a complete poset. We call $\circledast$ *absorbent-continuous* if

$$\text{for } x \in X^-,\ A(x)' \circledast x = x \text{ holds.} \tag{3}$$

The main result of this paper is a classification of absorbent-continuous, sharp FL_e-algebras on weakly real chains:

Theorem 5 *$\mathcal{U}$ is an absorbent-continuous, sharp FL_e-algebra on a weakly real chain if and only if its negative cone is a BL-algebra with components which are either cancellative or MV-algebras with two elements, and with no two consecutive cancellative components, its positive cone is the dual of its negative cone with respect to $'$, and its monoidal operation is given by (2).*

Absorbent-continuity can not be dropped from the conditions of Theorem 5, as shown by Example 6.

Example 6 Let $\mathbf{R}^* = \langle R \cup \{\bot, \top\}, +, 0 \rangle$ be the ordered abelian group of the reals added with two new elements $\bot$ and $\top$ as follows: Let $\bot < x < \top$ for $x \in R$. We extend the sum of the reals to $R \cup \{\bot, \top\}$ by letting $\bot + x = x + \bot = \bot$ for all $x \in R \cup \{\bot, \top\}$ and $x + \top = \top + x = \top$ for $x \in R$. We also extend the operation $-x$ to $\mathbf{R}^*$ by letting $-\top = \bot$ and $-\bot = \top$. Note that $\mathbf{R}^*$ is an ordered monoid and $-$ is an order reversing involution. Now denote Q the set of rational numbers, Q^* the set of irrational numbers, and let A be the set of all pairs $(a, b) \in (R \cup \{\bot, \top\}) \times (R \cup \{\bot, \top\})$ such that:

i. If either $a = \top$ or $a \in Q^*$, then $b = \bot$.

ii. If $a = \bot$, then $b = \top$.

In other words, A consists of $(\top, \bot)$, $(\bot, \top)$, of all $(a, \bot)$ such that a is not rational, and of all (a, b) such that a is rational and $b \in R \cup \{\bot, \top\}$. We order A lexicographically, i.e., $(a, b) \preceq (c, d)$ if either $a < c$ or $a = c$ and $b \leq d$. We further define a monoid operation $\circledast$ on A componentwise, i.e., $(a, b) \circledast (c, d) = (a + c, b + d)$.
Then we can show that $\langle A, \circledast, \leq, (0,0), (0,0)\rangle$ is a sharp FL_e-algebra on a weakly real chain. However, $\circledast$ is not absorbent-continuous and the negative cone of $\mathbf{A}$ is not a BL-algebra.
Finally, let us note that the algebra $\mathbf{A}$ being complete, order-dense, and separable, is isomorphic to a sharp uninorm algebra on $[0, 1]$, and its negative cone is not a BL-algebra, that is, the uninorm is not continuous in the negative cone. ■

Finally, we will show that closed intervals of real numbers are not the only example weakly real chains.

Theorem 7 *There is a complete and order-dense set X with maximum $\top$ and minimum $\bot$ such that:*

1. *X has a dense subset Y with $|Y| < |X|$.*

2. *For every $a, b < \top$ there are c, d such that $a < c < \top$, $b < d < \top$ and there exists a strictly increasing function from $[a, c]$ to $[b, d]$.*

3. *X is not isomorphic to $[0, 1]$.*

Without detailed proof we give the example below:

Case (a) Cantor's Continuum Hypothesis does not hold. Let

$$X = ((\aleph_1 \times [0, 1]) \cup \{\top\}) \setminus \{(\alpha + 1, 0) : \alpha < \aleph_1\}.$$

Order X with the lexicographic order: $(\alpha, a) \leq (\beta, b)$ if either $\alpha < \beta$ or $\alpha = \beta$ and $a \leq b$. Moreover, we stipulate that $\top$ is the top of X.

Case (b). Cantor's Continuum Hypothesis holds. Let $S = 2^{\aleph_1}$ be the set of all binary sequences of length $\aleph_1$, that is, the set of all functions from $\aleph_1$ into $\{0, 1\}$. Let $S_1 = \{s \in S : \exists \alpha < \aleph_1 (s_\alpha = 0 \wedge \forall \beta > \alpha (s_\beta = 1))\}$, and let $\top$ and $\bot$ be the sequences which are constantly 1 and constantly 0, respectively. Let $X = S \setminus S_1$. We order X by the lexicographic order. In other words, if $s \neq t$, there is a minimum α such that $s_\alpha \neq t_\alpha$. Then $s < t$ iff $s_\alpha < t_\alpha$.

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