A new representation theory: representing cylindric-like algebras by relativized set algebras

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As is known, cylindric algebras are not representable in the classical sense (as a subdirect product of cylindric set algebras), in general (see [He-Mo-Ta]). But, the Resek–Thompson theorem states that if the system of cylindric axioms is extended by a new axiom, by the merry-go-round (MGR, for short) property (furthermore axiom (C4) is weakened - only the commutativity of the single substitutions is assumed), then the cylindric–like algebra obtained is representable by a cylindric relativized set algebra (in particular, by an algebra in D_α, see [An-Th] and [Fe12a]). By an r-representation of a cylindric–like algebra we mean a representation by a cylindric-like *relativized* set algebra.

Analyzing the merry-go-round property, it turns out that this notation is underlain by the elementary operation of *transposition* (see [Fe11]).

The operation transposition cannot be introduced in every cylindric algebra. This fact led to research into the representability of transposition algebras (TA_α). Transposition algebras are cylindric algebras extended by abstract transposition operations (p_{ij}) and single substitutions (s^i_j), where $i, j < \alpha$, and they are weakening of the (so-called) finitary polyadic equality algebras introduced in [Sa-Th]. They are definitionally equivalent to the so-called non-commutative quasi-polyadic equality algebras. Transposition algebras are not necessarily representable in the classical sense. However, it is proven that transposition algebras are r-representable and they are representable by relativized set algebras in Gwp_α (see [Fe12a]).

The next question is that whether polyadic equality algebras are r- representable? As is known, the behaviour of polyadic algebras without equality (PA_α) and with equality (PEA_α) is also essentially different. Recall that polyadic algebras are essentially different from quasi-polyadic algebras in that the substitution operations are defined for real infinite transformations in the case of polyadic algebras. In [Fe12b], the problem of r-representability of polyadic equality algebras is answered for a large class: for polyadic equality algebras having ordinary cilindrifications (single cylindrifications), called *cylindric polyadic equality algebras* (class CPE_α). It is proven that this class is r -representable by algebras in $\mathsf{Gp}^{\mathrm{reg}}_\alpha$. Furthermore, Halmos's result on the representability of locally finite quasi-polyadic algebras ([Ha]) can be generalized for m-quasi, locally–m cylindric polyadic algebras (m is infinite) and r-representability.

The representant structures (representant set algebras) related to the above r-representation are very simple. For example, the representant of a *transposition algebra* (TA_α) is a Boolean set algebra with unit V such that V is an arbitrary union of weak Cartesian products, i.e., $V = \bigcup\limits_{k \in K} {}^{\alpha}U_k^{(p_k)}$ (class Gwp_α). Approaching our topic from set theory or geometry, the r-representation theorem says that the above Boolean set algebras are first order finite schema axiomatizable and the axioms can be the TA_α axioms. As is known if the disjointness of the members ${}^{\alpha}U_k^{(p_k)}$ is assumed in the above decompositions of V's, then the classes of set algebras obtained are no longer first order finite schema axiomatizable. Some additional non first order conditions for the algebra are needed (for example, the condition local finiteness). The representation theorems can be considered as an immediate generalizations of the Stone representation theorem for Boolean algebras.

A remarkable consequence of the investigation of the concept of r- representability is that *certain modifications* of the classical structures of algebraic logic came into the focus of research. A common feature of the abstract algebras occurring in these theorems is that the commutativity of cylindrifications is not required. Beyond this, additional axioms are assumed (e.g., the MGR property for cylindric algebras) or certain axioms are weakened (e.g., in the case of polyadic–style equality algebras, the last non-equality axiom).

The topic above is in a close connection with the neat embeddability problem, too (see [Fe10])

References

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