An axiomatic foundation of relativistic spacetime

Axiomatizations of the General Theory of Relativity.have occasionally been conducted to achieve precision and clarification of underlying concepts. Rigorous and inspiring work towards that aim based on first-order logic has been performed by Professor Németi and his working group (see references below). A complementary approach (Benda 2008) constructs the relativistic spacetime manifold ab initio in a theory ST, which is a conservative extension of Zermelo's Z with urelemente, interpreted as worldlines. The present work contains a few modifications and continues towards the construction of a metric, from where the Einstein tensor is readily built.

Two ideas characterize the present endeavour towards axiomatizing an important part of physics. First, there is a unified ontology of mathematics and physics, the set-theoretical hierarchy with urelemente. Mathematical objects are pure sets, which arise from the empty set by power set formation and union. (Accordingly, numbers are not mathematical objects. Rather, being a number is a property, instantiated, e.g., by the von Neumann numbers.) Physical objects are all sets that arise from the empty set and a set of urelemente by power set formation and union and are not mathematical objects. Secondly, physics is to be geometrized. Thus the urelemente are interpreted as geometrical entities, not, however, as spacetime points, but as worldlines. This is work in progress which is restricted to a part of physics. Particularly that crucial component of physics, experience, does not yet enter the present considerations.

To accommodate urelemente, the extension axiom of Z is modified and one axiom is added, stating that the urelemente form a set. In the language of ST, there are no individual constants and only two predicate constants, read "_is element of_" and "_intersects with_at_". 14 proper axioms of ST are given in five groups, which roughly state in terms of the intended interpretation: Intersection is of a worldline with another worldline at a real number; it is non-reflexive and between any two worldlines occurs mutually. For any two given worldlines, further worldlines exist, connecting them between certain sections thereof, while excluding certain inaccessible sections. Any given local region is densely filled by worldlines which do not intersect there. Of any two given worldlines, mutually inaccessible sections smoothly co-vary. Points on each worldline are determined by therefrom inaccessible sections of two other worldlines. In spite of the suggestive language of the foregoing rough description in terms of the intended interpretation, no underlying space, in particular, no metric is presupposed.

The axioms are motivated by interpreting worldlines as possible particle paths. Yet they are guided by formal deliberations, as well. To see this, we paraphrase the axioms of ST more accurately, albeit, for brevity, still in an incomplete manner. Emphasizing a formal motive for the axioms, we write "relates to" for "intersects" in the next paragraph. If a first object relates to a third object at some first real number and a second object relates to the third object at some second real number, we will temporarily say that the first object “indirectly relates to” the second object at the first and second real numbers. In that case, we say that the second object at the second real number is “connectable” from the first object at the first real number. For any first object at a first real number, we also temporarily speak of a first real number “attached” to a first object. Symmetry of indirect relating is not presumed, so we speak of “backward” indirect relating and its converse, “forward” indirect relating, as well as of “backward connectable” and “forward connectable” real numbers attached to objects. We speak of a “dense” set of objects, if their relating to some further object occurs at all real numbers of some open interval of real numbers. What is called a “diamond” is a cartesian product of a set of urelemente and a set of real numbers, which are respectively backward and forward connectable from first and second real numbers that are attached to some guiding urelement.

With that, our informal paraphrasing proceeds as follows. A ternary predicate constant, “relates to”, has only arguments the first and third of which are urelemente and the second of which is a real number, which is a pure set (P1). Relating is irreflexive in its urelement arguments (P2). No urelement relates to a second urelement more than finitely many times within any finite interval of real numbers (P3). Relating is not formally symmetric in its urelement arguments, yet swapping these again yields their relating at some real number (P4). A three-party symmetry of relating holds: If of three given urelemente two relate to the third at the same real number, then any two of those three urelemente relate to the remaining third urelement at a common real number (P4). For any two urelemente, another urelement exists, which relates to them indirectly, from a given first real number attached to a first urelement backward to some second real number attached to the second urelement (P5). From any first urelement, any second urelement at at least two real numbers is not connectable (P6). If two urelemente exist that relate to each other, they also relate indirectly to each other via more than countably many urelemente at pairs of real numbers in determined ranges (P7), (P8), (P9). Every pair of real numbers and urelemente attached thereto is contained in some diamond whose elements have dense first components (P10). Changing any first real number attached to a first urelement shifts the infimum of the set of not connectable second real number attached to a second urelement smoothly and monotonously (P11). Any set of real numbers on a second urelement that is not connectable from a given first real number attached to a first urelement is, after undergoing a linear shift, again a not connectable set from some real number attached to some urelement (P12). Not connectable sets of real numbers on any two urelemente are locally unique in that precisely all their elements are inaccessible from at most one pair of a real number and a urelement out of some diamond containing that pair (P13).

Thus the axioms draw more from formal than from obviously physical considerations They are ontologically and conceptually parsimonious. The only primitive physical entities are worldlines. The only high-level mathematical entities mentioned are real numbers and smooth real functions, which are constructible from the frame theory Z.

Spacetime points are non-primitive, they are defined as sets of mutually intersecting worldlines and in turn form a set stp. Worldlines can now be visualized as connecting spacetime points, and each spacetime point on a given worldline is characterized by a unique real number. An Alexandroff topology is introduced. With that, to aid visualization further, a lightcone around a spacetime point *p* is definable as the set of topological boundary points of the set of spacetime points containing some common worldline with *p*. By the axioms of ST, spacetime points form a four-dimesional manifold, which is oriented and Haussdorf. Around each spacetime point *p*, a canonic coordinate system is found, mapping spacetime points in a neighborhood of *p* to a quadruple of real numbers that characterize the intersection points of two worldlines with the lightcone around *p*. All canonic coordinate systems form a unique atlas, a "standard spacetime atlas". Later definitions of smooth functions involving coordinates will be carried out with respect to the standard spacetime atlas.

From here, it is straightforward, if involved, using no more than the present theory ST and its frame Z, to define entities known from differential geometry and prove their properties: smooth real functions on spacetime, which form a set fst; curves; tangent vector fields, which form a set vst; connections; and tensors. Tangent vector fields are mappings on fst. Being fst-linear, they form a ring over the module of spacetime functions, having bases that are derivatives. Transformations of tangent vector fields are defined as those that preserve the chain rule of derivatives. Velocity tangent vectors along curves are defined as images of such transformations from derivatives on one-dimensional manifolds. Compositions of pairs of tangent vector fields define mappings from vst $\times$ vst to vst. Each of them is fst-linear in the first argument and $\mathbb{R}$-linear in the second argument. Connections are now defined as analogues of such compositions, as mappings from vst $\times$ vst to vst with the same linearity properties. That is in accordance with the common definition. However, here no metric is yet available to specify connections further. For each connection, a thereby induced covariant derivative along a given curve is defined in the usual way. 02-tensors are defined as mappings from vst $\times$ vst to fst which are fst-bilinear, forming a set t02. Furthermore, from any given connection, a tensor derivation is constructed, that is, a set of $\mathbb{R}$-linear mappings on fst, vst and t02, which follow a product rule.

Spacetime points that contain a common worldline form curves called "worldline curves" (what are called "worldlines" in common parlance). By the axioms of ST, there exists precisely one connection d, such that all worldline curves are free of acceleration, that is, the d-induced covariant derivatives of the velocity vector fields along all worldline curves vanish. From the connection d, a unique tensor derivation, t, is constructed.

Only now, a metric g is defined: as a 02-tensor whose tensor derivation t vanishes. The metric is thereby, up to a constant g_0, determined by the worldlines intersecting each other at given real numbers. By postulating a Lorentz signature for g_0, the metric g has a Lorentz signature throughout spacetime.

Once the metric is in place, the Einstein tensor is obtained via a famous construction by Lovelock (1971), which was improved by Navarro (2010). It is determined up to a constant g_0 times the scalar curvature s. This indeterminacy differs from the cosmological constant by the factor s. That may provide a testable condition.

It is in the spirit of the present geometric approach to view the right-hand term in the Einstein equation, denoting the energy-momentum tensor, as being defined. With that, ST needs an extension, postulating a condition to make energy-momentum empirically accessible.

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