Interpretation of Special Relativity in the Language of Newtonian Kinematics

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A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

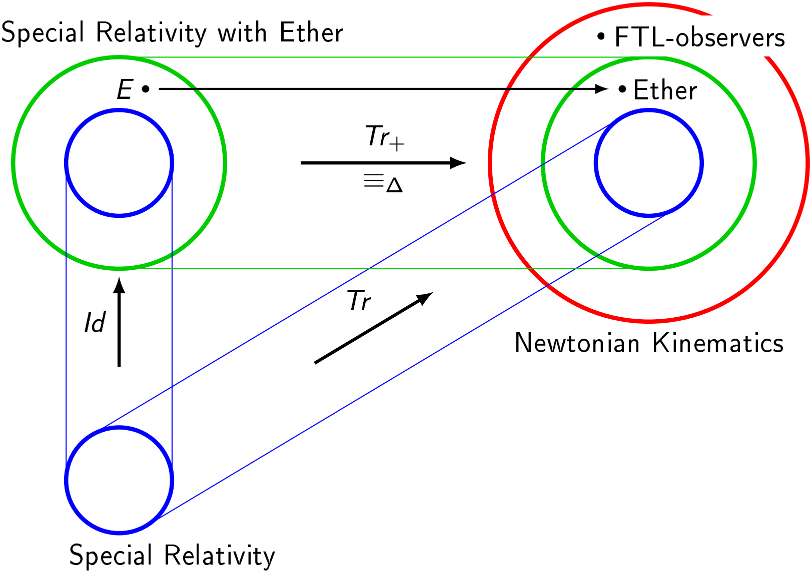
An *interpretation* of theory $\mathbf{Th_1}$ in theory $\mathbf{Th_2}$ is a translation Tr which translates all axioms of $\mathbf{Th_1}$ into theorems of $\mathbf{Th_2}$.

An interpretation is a *definitional equivalence* if a translated formula can be translated back such that it becomes a formula which is equivalent to the original formula.

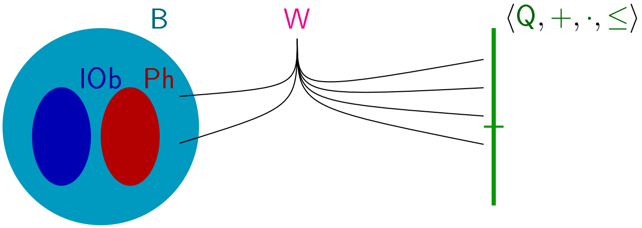
We use these concepts to show the *differences* between theories which are *not* equivalent.

There are translations Tr, Tr_{+}, and Tr_{+}^{-1} between the languages of NK and SR such that:

- $NK \vdash Tr(SR)$
- $NKnoFTL \vdash Tr_{+}(SRwithEther)$
- $SRwithEther \vdash Tr_{+}^{-1}(NKnoFTL)$
- Definitional equivalence: $SRwithEther \equiv_{\Delta} NKnoFTL$, i.e., Tr_{+} and Tr_{+}^{-1} are inverses of each other up to logical equivalence in $NKnoFTL$ and $SRwithEther$.


Special Relativity with Ether
FTL-observers
E
Ether
Tr_+
$\equiv_\Delta$
Id
Tr
Newtonian Kinematics
Special Relativity

Language: $\{ \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W} \}$



B ⇝ Bodies (things that move)

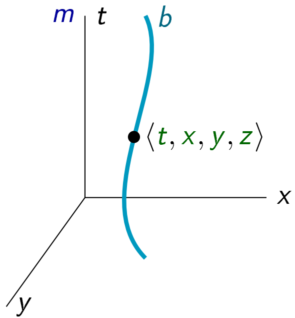
IOb ⇝ Inertial Observers   Ph ⇝ Photons (light signals)

Q ⇝ Quantities

$+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

W ⇝ Worldview (a 6-ary relation of type BBQQQQ)

$W(m, b, t, x, y, z) \leftrightsquigarrow$ " observer m coordinatizes body b at spacetime location $\langle t, x, y, z \rangle$."



Worldline of body b according to observer m

$$wl_m(b) := \{\langle t, x, y, z\rangle \in Q^4 \ : \ W(m, b, t, x, y, z)\}$$

$$\textbf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}\}$$

$$\textbf{NewtonianKin}_{\textit{Full}} := \textbf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\uparrow}_{+}\}$$

$$\textbf{SpecRel}_{\textit{Full}} := \textbf{Kin} \cup \{\mathsf{AxPh}_c, \mathsf{AxThExp}^{\uparrow}\}$$

AxEField :

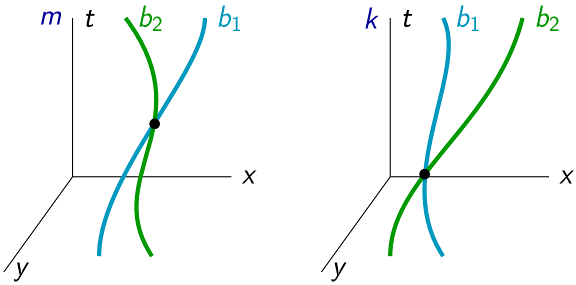
The **structure of quantities** $\langle Q, +, \cdot, \leq \rangle$ *is an Euclidean field,*

- Real numbers: $\mathbb{R}$,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- Hyperreal numbers: $\mathbb{R}^*$,
- Real constructable numbers,
- Etc...

AxEv :

Inertial observers coordinatize the same events (meetings of bodies).

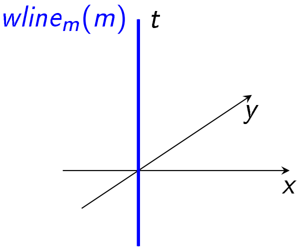
$$ev_m(\bar{x}) := \{b : W(m, b, \bar{x})\}$$



$$\forall m\, k\bar{x} \;\; [\mathsf{IOb}(m) \land \mathsf{IOb}(k) \rightarrow \exists \bar{y} \;\; ev_m(\bar{x}) = ev_k(\bar{y})].$$

AxSelf :

Every Inertial observer is stationary according to himself.

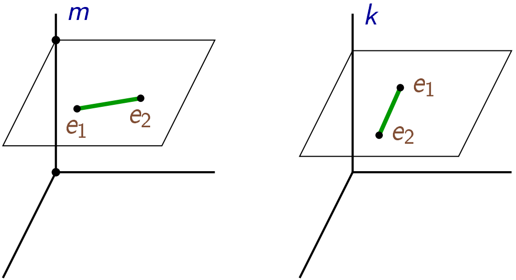


$$\forall mtxyz \ \Big(\mathsf{IOb}(m) \rightarrow \big[\mathsf{W}(m, m, t, x, y, z) \leftrightarrow x = y = z = 0\big]\Big).$$

AxSymD :

Inertial observers agree as to the *spatial distance* between two *events* if these two *events* are *simultaneous* for both of them.

$$\mathrm{space}(\bar{x},\bar{y}) := \sqrt{(x_2-y_2)^2+\ldots+(x_d-y_d)^2}$$



$$\forall m k \bar{x}\bar{y}\bar{x}'\bar{y}' \ \ [IOb(m) \land IOb(k) \land x_1 = y_1 \land x_1' = y_1' \land ev_m(\bar{x}) = ev_k(\bar{x}') \land ev_m(\bar{y}) = ev_k(\bar{y}') \to \mathrm{space}(\bar{x},\bar{y}) = \mathrm{space}(\bar{x}',\bar{y}')]$$

AxLine :

The worldlines of inertial observers are straight lines according to inertial observers.

$$\forall m k \bar{x} \bar{y} \bar{z} \ \big(IOb(m) \land IOb(k) \land W(m,k,\bar{x}) \land W(m,k,\bar{y}) \land W(m,k,\bar{z}) \\ \rightarrow \exists a \ [\bar{z}-\bar{x} = a(\bar{y}-\bar{x}) \lor \bar{y}-\bar{z} = a(\bar{z}-\bar{x})]\big).$$

(In ***SpecRel***, AxLine is a theorem, so including it as an axiom is redundant.)

AxTriv :

Any trivial transformation of an inertial frame is also an inertial frame.

$\forall T \in Triv[\forall m \exists k(w_{mk} = T)]$, where Triv is the set of Trivial transformations, i.e. transformations that are isometries on space and translations on time.

AxAbsTime :

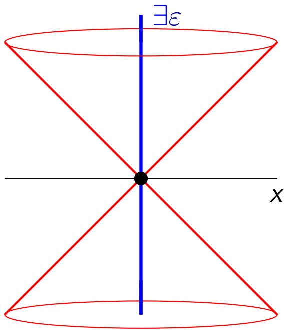
The time difference between two events is the same for all inertial observers.

$$\mathrm{time}(\bar{x},\bar{y}) := |x_1 - y_1|$$

$$\forall m k \bar{x}\bar{y}\bar{x}'\bar{y}' \;\; [IOb(m) \wedge IOb(k) \wedge ev_m(\bar{x}) = ev_k(\bar{x}') \wedge ev_m(\bar{y}) = ev_k(\bar{y}') \rightarrow \mathrm{time}(\bar{x},\bar{y}) = \mathrm{time}(\bar{x}',\bar{y}')].$$

AxEther(Einstein's AxLight) :

There exists an inertial observer in which the light cones are right.

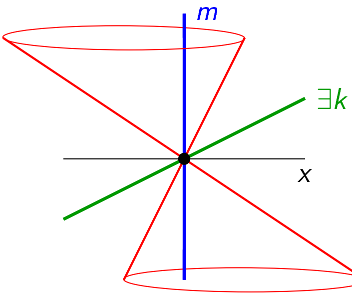


$$\exists \varepsilon c \Big[\mathsf{IOb}(\varepsilon) \land c > 0 \land \forall \bar{x}\bar{y} \Big(\exists p \big[\mathsf{Ph}(p) \land \mathsf{W}(\varepsilon, p, \bar{x}) \land \mathsf{W}(\varepsilon, p, \bar{y}) \big] \leftrightarrow \mathsf{space}(\bar{x}, \bar{y}) = c \cdot \mathsf{time}(\bar{x}, \bar{y}) \Big) \Big]$$

$\mathsf{AxThExp}_{+}^{\uparrow}$:

Inertial observers can move along any non-horizontal straight line.

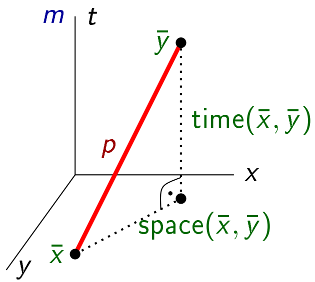
$$k \uparrow m \overset{def}{\iff} ev_k(\bar{x}) = ev_m(1,0,0,0) \wedge ev_k(\bar{y}) = ev_m(0,0,0,0) \rightarrow x_1 > y_1$$



$$\exists h \ \big(IOb(h)\big) \wedge \forall m\bar{x}\bar{y} \ \big(IOb(m) \wedge x_1 \neq y_1$$
$$\rightarrow \exists k \ \ IOb(k) \wedge W(m,k,\bar{x}) \wedge W(m,k,\bar{y}) \wedge m \uparrow k\big).$$

$\mathsf{AxPh_c}$:

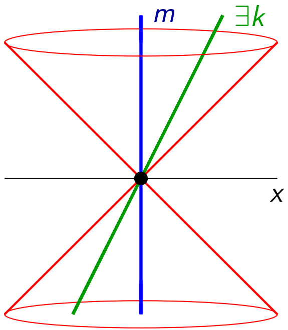
For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.



$$\exists c\Big[c > 0 \land \forall m\bar{x}\bar{y}\Big(\mathsf{IOb}(m) \to \exists p\big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y})\big] \leftrightarrow \mathsf{space}(\bar{x},\bar{y}) = c\cdot\mathsf{time}(\bar{x},\bar{y})\Big)\Big]$$

AxThExp$^{\uparrow}$:

Inertial observers can move with any speed slower than that of light.



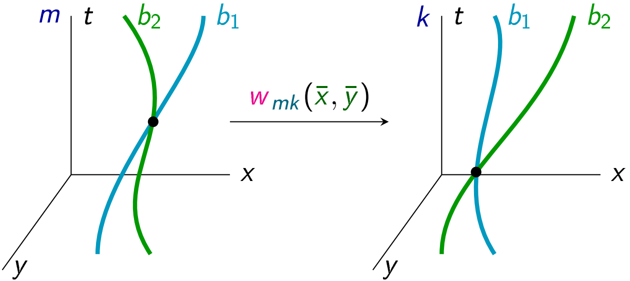
$$\exists h \ \ [IOb(h)] \wedge \forall m\bar{x}\bar{y} \ \ \big(IOb(m) \wedge \mathrm{space}(\bar{x},\bar{y}) < c \cdot \mathrm{time}(\bar{x},\bar{y}) \\ \rightarrow \exists k \ \ [IOb(k) \wedge W(m,k,\bar{x}) \wedge W(m,k,\bar{y}) \wedge m \uparrow k]\big).$$

$$\textbf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}\}$$

$$\textbf{NewtonianKin}_{\textit{Full}} := \textbf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\uparrow}_{+}\}$$

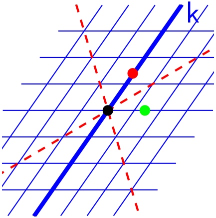
$$\textbf{SpecRel}_{\textit{Full}} := \textbf{Kin} \cup \{\mathsf{AxPh}_c, \mathsf{AxThExp}^{\uparrow}\}$$

Worldview transformation :

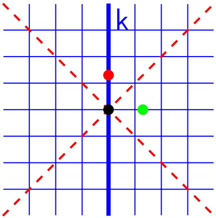


$$w_{mk}(\bar{x}, \bar{y}) \overset{def}{\iff} ev_m(\bar{x}) = ev_k(\bar{y})$$

Galilean transformations:

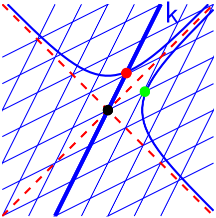


worldview of m

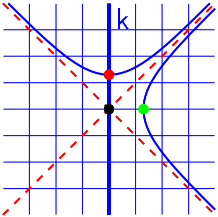


worldview of k

Poincaré transformations:



worldview of m



worldview of k

Representation Theorems:

Theorem:

$SpecRel_{Full} \vdash \forall mk[IOb(m) \wedge IOb(k) \rightarrow$ " w_{mk} *is a Poincaré Transformation*"].

- ca. 1998 proven for a version of BasAx strongly related to SpecRel in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi
- Synthese 2012 "A logic road from special relativity to general relativity" by H. Andréka, J. X. Madarász, I. Németi & G. Székely (Theorem 2.2)

Theorem:

$NewtonianKin_{Full} \vdash \forall mk[IOb(m) \wedge IOb(k) \rightarrow$ " w_{mk} *is a Galilean Transformation*"].

- proof for a different version of Newtonian Kinematics in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi

Theorem:

$\mathit{SpecRel}_{\mathit{Full}} \vdash \forall k P[IOb(k) \wedge$ "P is a (orthochronous) Poincaré Transformation" $\rightarrow \exists k' w_{kk'} = P]$.

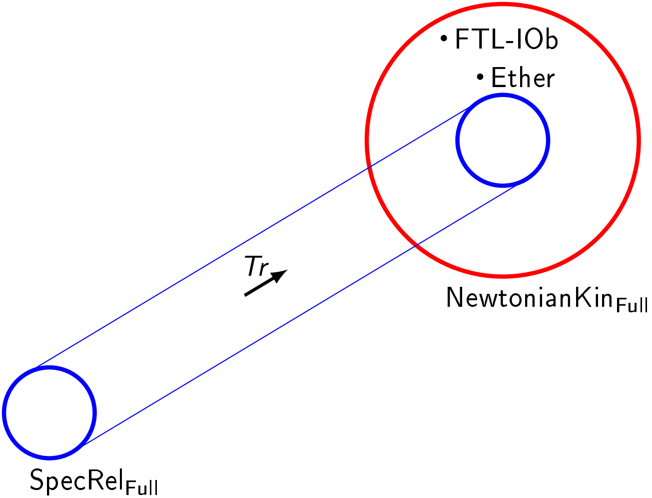
Theorem:

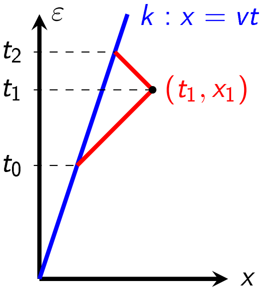
$\mathit{NewtonianKin}_{\mathit{Full}} \vdash \forall k G[IOb(k) \wedge$ "G is a (orthochronous) Galilean Transformation" $\rightarrow \exists k' w_{kk'} = G]$.

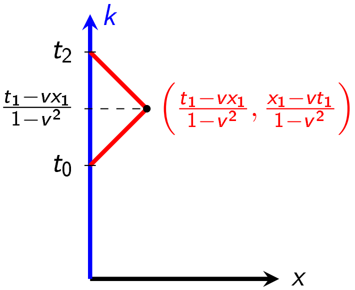
- can be proven based on the ideas in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi

Theorem:

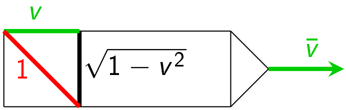
There is an interpretation Tr of **SpecRel$_{Full}$** *in* **NewtonianKin$_{Full}$**.

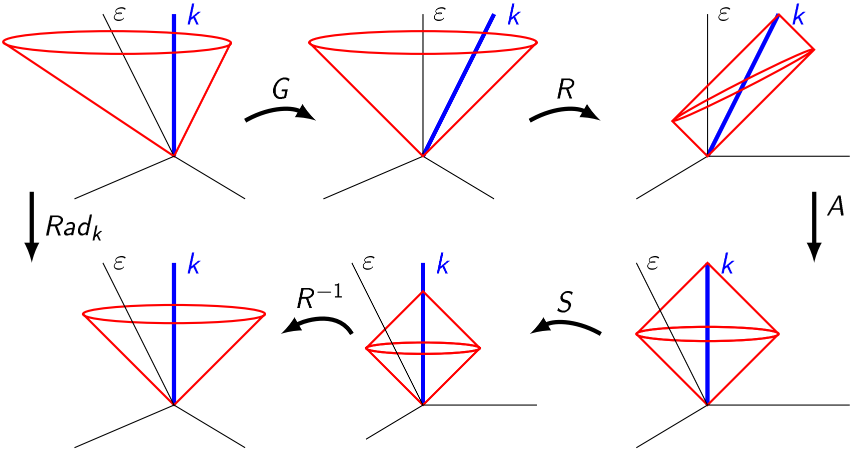






$$A = \begin{bmatrix} \frac{1}{1-v^2} & \frac{-v}{1-v^2} & 0 & 0 \\ \frac{-v}{1-v^2} & \frac{1}{1-v^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-v^2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{1-v^2}} \end{bmatrix} \qquad S = \sqrt{1 - v^2} \quad (\text{if } c = 1)$$





$$Rad_k = R^{-1} \circ S \circ A \circ R \circ G$$

$$Tr(a+b=c) := a+b=c$$

$$Tr(a\cdot b=c) := a\cdot b=c$$

$$Tr(a<b) := a<b$$

$$Tr\big(\mathsf{W}_{SR}(k,b,t,x,y,z)\big):= \exists t'x'y'z'[\mathsf{W}_{NK}(k,b,t',x',y',z')\wedge Rad_k(t',x',y',z')=(t,x,y,z)]$$

$$Tr\big(\mathsf{IOb}_{SR}(k)\big):=\mathsf{IOb}_{NK}(k)\wedge\forall\varepsilon[Ether(\varepsilon)\rightarrow speed^{NK}_{\varepsilon}(k)<c]$$

where

$$Ether(\varepsilon) \stackrel{def}{\Longleftrightarrow} \mathsf{IOb}_{NK}(\varepsilon)\wedge\forall p[Ph(p)\rightarrow speed^{NK}_{\varepsilon}(p)=c]$$

and

$$speed_m(k)=v \stackrel{def}{\Longleftrightarrow} \forall\bar{x},\bar{y}\in\mathrm{Q}^d[\mathsf{W}(m,k,\bar{x})\wedge\mathsf{W}(m,k,\bar{y})\wedge x_1\neq y_1 \rightarrow \frac{space(\bar{x},\bar{y})}{\mathrm{time}(\bar{x},\bar{y})}=v]$$

Example: Translation of $\mathrm{AxSelf}_{\mathrm{SR}}$

$$Tr[\forall m\ \mathrm{IOb}_{SR}(m) \rightarrow \forall \bar{x}\ (\mathrm{W}_{SR}(m,m,\bar{x}) \leftrightarrow x_2 = \ldots = x_d = 0)]$$

$$\equiv \forall m\ \left[\mathrm{IOb}_{NK}(m) \wedge \forall \varepsilon [Ether(\varepsilon) \rightarrow speed_{\varepsilon}^{NK}(m) < c]\right.$$

$$\left.\rightarrow \forall \bar{x}\ (\exists \bar{y}[\mathrm{W}_{NK}(m,m,\bar{y}) \wedge Rad_k(\bar{y}) = \bar{x}] \leftrightarrow x_2 = \ldots = x_d = 0)\right]$$

Theorem:

There is an interpretation Tr of $\mathbf{SpecRel}_{\mathit{Full}}$ *in* $\mathbf{NewtonianKin}_{\mathit{Full}}$.

$$\mathbf{NewtonianKin}_{\mathit{Full}} \vdash Tr(\mathbf{SpecRel}_{\mathit{Full}})$$

- $\mathbf{AxEField_{NK}} \vdash Tr(\mathbf{AxEField_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathrm{AxEther_{NK}}, \mathbf{AxEv_{NK}} \vdash Tr(\mathbf{AxEv_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathrm{AxEther_{NK}}, \mathbf{AxSelf_{NK}} \vdash Tr(\mathbf{AxSelf_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathrm{AxEther_{NK}}, \mathbf{AxSymD_{NK}} \vdash Tr(\mathbf{AxSymD_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathbf{AxLine_{NK}}, \mathrm{AxEther_{NK}} \vdash Tr(\mathbf{AxLine_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathrm{AxEther_{NK}}, \mathbf{AxTriv_{NK}} \vdash Tr(\mathbf{AxTriv_{SR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathbf{AxEther_{NK}} \vdash Tr(\mathbf{AxPh_{cSR}})$
- $\mathrm{AxEField_{NK}}, \mathrm{AxLine_{NK}}, \mathrm{AxEther_{NK}}, \mathbf{AxThExp^{\uparrow}_{+NK}} \vdash Tr(\mathbf{AxThExp^{\uparrow}_{SR}})$

$\mathsf{NewtonianKin}^{NoFTL}_{Full} := \mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\uparrow}_{\mathsf{NoFTL}}, \mathsf{AxNoFTL}\}$

$\mathsf{AxThExp}^{\uparrow}_{\mathsf{NoFTL}}$:

Inertial observers can move with any speed which is in the ether frame slower than that of light.

$$\exists h \ \big(IOb(h)\big) \land \forall \varepsilon \bar{x}\bar{y}\Big(Ether(\varepsilon) \land \mathrm{space}(\bar{x},\bar{y}) < c \cdot \mathrm{time}(\bar{x},\bar{y}) \rightarrow \exists k \ \ IOb(k) \land W(\varepsilon,k,\bar{x}) \land W(\varepsilon,k,\bar{y}) \land m \uparrow k\Big).$$

AxNoFTL :

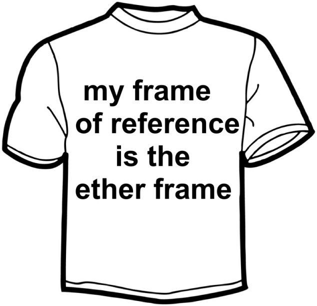
All inertial observers move slower than light with respect to the ether frames.

$$\forall m\varepsilon[IOb(m) \land Ether(\varepsilon) \rightarrow Speed^{NK}_{\varepsilon}(m) < c].$$

$\mathsf{SpecRel}^{\varepsilon}_{Full} := \mathsf{SpecRel}_{Full} \cup \{\text{AxPrimitiveEther}\}$

AxPrimitiveEther :

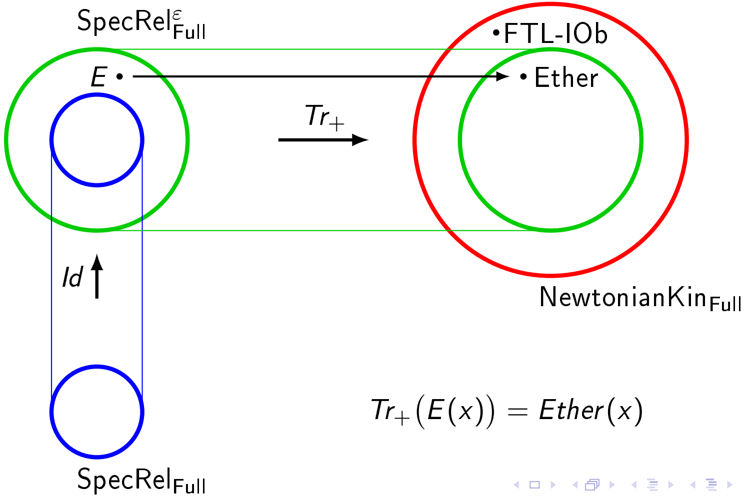
There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.



$$\exists\varepsilon\big(E(\varepsilon)\land\forall k\big[[IOb(k)\land(\exists T\in Triv)w^{SR}_{\varepsilon k}=T]\leftrightarrow E(k)\big]\big)$$

Theorem:

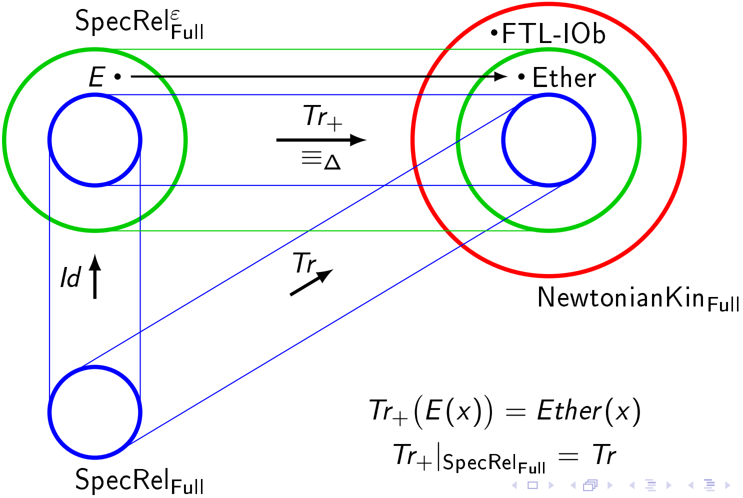
Tr_+ is a definitional equivalence between $\mathbf{SpecRel}^{\varepsilon}_{\mathit{Full}}$ *and* $\mathbf{NewtonianKin}^{\mathit{NoFTL}}_{\mathit{Full}}$.



$$Tr_+(E(x)) = Ether(x)$$

Theorem:

Tr_+ is a definitional equivalence between $\mathsf{SpecRel}^{\varepsilon}_{Full}$ *and* $\mathsf{NewtonianKin}^{NoFTL}_{Full}$.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathsf{SpecRel}_{\mathsf{Full}}} = Tr$$