What structures can numbers have in relativity theory?

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A main goal of foundational investigations is to get a clear understanding of all the fundamental assumptions of the investigated theory (or class of theories), as well as their roles and logical connections.

Some of the assumptions are easy to spot since they have central roles even in the non-axiomatic approaches, such as the principle of relativity or the light postulate. However, only few of the assumptions are so upfront in the informal theories of physics, most of them are hidden and it is difficult to even realize that they are assumed, not to mention to understand their roles in the theory. One of these not so upfront assumptions is that the structure of physical quantities is isomorphic to the structure of real numbers.

Almost all of the physical theories assume the existence of this isomorphism. However, this assumption is not at all self-evident since the outcomes of physical measurements are finite decimals. This fact suggest that the rational numbers (or even the integers) should be enough to model the theories of physics.

Therefore, the investigation of the role of our assumptions about the structure of quantities is a natural foundational research direction. A way to approach to this research direction is investigating the following question in the case of some fixed physical theory Th.

"Over which structures of quantities can theory Th be modeled?"

The most natural reason for this investigation is that we cannot experimentally verify whether the structure of physical quantities is isomorphic to the field of real numbers or not (the existence of such isomorphism cannot be confirmed by experiments). So we cannot support this assumption by empirically. Therefore, it is important to understand if the field of real numbers can or cannot be replaced with some other structure (such as the field of rational numbers).

Another natural reason for these investigation is that it may lead to a deeper understanding of the connections between the mathematical assumptions about the quantities and the other (physical) assumptions of the theory. In principle, the more general algebraic structures we can use to model the quantities the more flexible our theory becomes.

In this talk we are going to investigate this question only in the case of different axiomatic theories of relativity. However, the question can be asked and investigated in any theory of physics in the same way.

Our typical results in this area show that some physical axioms require extra properties on quantities, such as the existence of square roots of positive numbers.

For example, in SpecRel, one of our axiom systems for special relativity, we assume only that the structure of quantities is an ordered field. SpecRel together with axiom AxThExp which ensures the possibility of motion of inertial observers with any speed less than that of light imply that every positive quantity has a square root [1]. So axiom system SpecRel+AxThExp cannot be modeled over the field of rational numbers. Therefore, AxThExp can be a theoretical reason explaining why the structure of quantities cannot be the field of rational numbers.

However, with a slight adjustment of the axioms, special relativity can be modeled over the field of rational numbers. To do so, we should assume only that observers can move approximately with any speed less than that of light ($\mathsf{AxThExp}^-$) instead of AxThExp [4]. Since physical measurements are never perfectly accurate, axiom $\mathsf{AxThExp}^-$ is physically more natural than AxThExp.

What about general relativity?

In the axiomatic road toward general relativity there is a strong drifting towards the field of real numbers. For example, the continuity axiom scheme we needed to prove the twin paradox in [3] implies that the structure of quantities is elementarily equivalent to the field of reals. This means that it is indistinguishable from the reals in the language of ordered fields, i.e., $\{0, 1, +, \cdot, \leq\}$.

Moreover, if we assume that observers can accelerate uniformly, then it can be proved that the field of algebraic real numbers cannot be the structure of quantities [5]. Since the real numbers and the algebraic real numbers are elementarily equivalent, this implies that the class of those structures over which the corresponding axiom system of general relativity (or accelerated observers) can be modeled is not an elementary class (i.e., it is not axiomatizable within first-order logic in the language of ordered fields).

In it is also natural to assume that observers can move on any definable timelike (i.e., slower than light) curve. This assumption makes even more restrictions on the possible structures of quantities [2].

In this talk we will also mention some of the several open problems of this research direction.

References

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