Splitting methods in algebraic logic in connection to non-atom–canonicity and non-first order definability

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Abstract . We deal with various splitting methods in algebraic logic. The word 'splitting' refers to splitting some of the atoms in a given relation or cylindric algebra each into one or more subatoms obtaining a bigger algebra, where the number of subatoms obtained after splitting is adjusted for a certain combinatorial purpose. This number (of subatoms) can be an infinite cardinal. The idea originates with Leon Henkin. Splitting methods existing in a scattered form in the literature, possibly under different names, proved useful in obtaining (negative) results on non–atom canonicity, non–finite axiomatizability and non–first order definability for various classes of relation and cylindric algebras. In a unified framework, we give several known and new examples of each. Our framework covers Monk's splitting, Andréka's splitting, and, also, so–called blow up and blur constructions involving splitting (atoms) in finite Monk–like algebras and rainbow algebras.

Henceforth, we follow the notation of [2] which is in conformity with the notation of [4]. Besides cylindric algebras CA, we deal with the following cylindric–like algebras Sc (Pinter's substitution algebras) and $\mathsf{QA}(\mathsf{QEA})$ quasi–polyadic (equality) algebras. For K any of these classes and α any ordinal, we write K_α for the variety of α–dimensional K algebras, and $(\mathsf{C})\mathsf{RK}_\alpha$ for the class of (completely) representable K_αs. For an ordinal α, $(\mathsf{R})\mathsf{Df}_\alpha$ denotes the class of (representable) diagonal free CA_αs. For a class $\mathbf{K}$ of Boolean algebras with operators we write $\mathbf{K} \cap \mathbf{At}$ for the class of atomic algebras in $\mathbf{K}$.

Fix $2 < n < \omega$. The idea of splitting one or more atoms to subatoms in an algebra to get a (bigger) superalgebra tailored to a certain purpose originates with Henkin [4, p.378, footnote 1]. In the cylindric paradigm, Andŕeka modified such splitting methods *re-inventing* (Andréka's) splitting. In this new setting, Andréka proved a plethora of *relative non–finite axiomatizability results* [1] like for e.g RQEA_n is not finitely axiomatizable over RQA_n nor RCA_n. In the former case Andréka went further excluding universal axiomatizations containing only finitely many variables, a result that we lift to the transfinite below.

Though splitting techniques are associated more in the literature with non–finite axiomatizability results, in this paper we argue and indeed demonstrate that there are several subtle re–incarnations of this technique proving results on notions like *non–atom canonicity* (to be defined below) and *non–finite first order definability.*

1 A variation on Andréka's (most famous) splitting:

We let $\mathfrak{Rd}_{ca}$ denote 'cylindric reduct' and $\mathfrak{Rd}_{qa}$ denote 'quasi—polyadic reduct.' We show that the variety RQEA_ω cannot be axiomatized by a set of universal formulas containing finitely many variables over RQA_ω. The proof is an instance of Andréka's splitting [1]. For each positive k we construct $\mathfrak{A} \notin \mathsf{RQEA}_\omega$ such that its k- subalgebras (subalgebras generated by at most k–elements) are in RQEA_ω, $\mathfrak{Rd}_{ca}\mathfrak{A} \notin \mathsf{RCA}_\omega$ and $\mathfrak{Rd}_{qa}\mathfrak{A} \in \mathsf{RQA}_\omega$. But for fixed k not only one splitting is done, but infinitely many each (to an atom) in a different set algebra; the resulting algebras (obtained after splitting) form a chain; their directed union will be the $\mathfrak{A}$ we want. This can (and will) be done for each positive k. Accordingly, throughout the proof fix a positive k.

Proof. **(1) Splitting a single atom to finitely many subatoms getting non–representable algebras from representable ones:** For fixed $2 < n < \omega$, take a finite $m \geq 2^{k\times n!+1}$. Suppose that the signature consists of ω–many cylindrifier, $\mathsf{c}_i : i < \omega$, diagonal constants $\mathsf{d}_{ij}, i < j < \omega$, and $n^2 - n$ substitutions $\mathsf{s}_{[i,j]} : i < j < n$. One forms, for each such n, an algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ in this specified signature, by splitting the ω–ary relation $R = \prod_{i\in\omega} U_i$ with $U_0 = m - 1$ and $|U_i| = m$ for $0 < i < \omega$ in the algebra $\mathfrak{A}_n = \mathfrak{Sg}^{\wp({}^\omega U)}\{R\}$, where $U = \bigcup_{i\in I} U_i$ into m abstract copies. Observe that here R depends on n, because m depends on n and R depends on $U_0 = m - 1$. The resulting algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ therefore has the signature expanding CA_ω by the finitely many substitution operators $\mathsf{s}_{[i,j]}$, $i < j < n$. Here the set algebra $\wp({}^\omega U)$ is taken in the specified signature with operations interpreted the usual way as in set algebras, e.g. $\mathsf{S}_{[0,1]}\{R\} = \{s \in {}^\omega U : (s_1, s_0) \in R\}$. It can be easily checked that for all $i < j < n$, $\mathsf{S}_{[i,j]}R$ is an atom in $\mathfrak{A}_n$. In particular, R is partitioned into a family $(R_i : i < m)$ of atoms in the bigger algebra $\mathsf{split}(\mathfrak{A}_n R, m)(\supseteq \mathfrak{A}_n)$, so that $R = \sum_{i<m} R_i$, where $m = |U_0| + 1$. Furthermore in $\mathsf{split}(\mathfrak{A}_n, R, m)$, we have $\mathsf{s}_{[l,j]}R = \sum_{i<m} \mathsf{s}_{[l,j]}R_i$ and $\mathsf{c}_t\mathsf{s}_{[l,j]}R_i = \mathsf{c}_t\mathsf{s}_{[l,j]}R$ for all $l, j < n$, $i < m$ and $t < \omega$; so that, in particular, R is *cylindrically equivalent* to its abstract copies. The algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ is determined uniquely (up to isomorphism) by $\mathfrak{A}_n, R$ and m, hence the notation, and it will not be representable. Even more, the algebra $\mathfrak{Rd}_{ca}\mathsf{split}(\mathfrak{A}_n, R, m)$ having CA_ω signature will not be representable for the following reasoning: One defines the term $\tau(x) = (\bigwedge_{i<m} \mathsf{s}^0_i\mathsf{c}_1 \ldots \mathsf{c}_m x \cdot \bigwedge_{i<j<n} -\mathsf{d}_{ij})$ as in [1, Top of p.157]. Then $\mathfrak{A}_n \models \tau(R) = 0$ hence $\mathsf{split}(\mathfrak{A}_n, R, m) \models \tau(R) = 0$ because $\mathfrak{A}_n \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$. Identifying set algebras with their domain, for an algebra $\mathfrak{A}$ and a non–zero $a \in \mathfrak{A}$, we say that a representation $h : \mathfrak{A} \to \wp({}^\omega U)$ *respects the non–zero element* a if $h(a) \neq \emptyset$. If $\mathsf{split}(\mathfrak{A}_n, R, m)$ were representable, then it will have a representation that respects R. But any such representation h will satisfy that $\tau(h(R)) \neq 0$ which is impossible.

(2) Representability of k–generated subalgebras: Now we show that the k–generated subalgebras are representable. Let $G \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$, $|G| \leq k$. Let $\mathfrak{P} = (R_l : l < m)$ be the abstract partition of R in the bigger algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ obtained by splitting R in $\mathfrak{A}_n$ into m (abstract) subatoms $(R_l : l < m)$. One defines the following relation on $\mathfrak{P}$: For $l, t < m$, $R_l \sim R_t \iff (\forall g \in G)(\forall i, j < n)(\mathsf{s}_{[i,j]}R_l \leq g \iff \mathsf{s}_{[i,j]}R_t \leq g)$. Then it is straightforward to check that $\sim$

is an equivalence relation on $\mathfrak{P}$ having $p < m$ many equivalence classes, because $|G| \leq k$, $n^2 - n < n!$ and (recall that) $m \geq 2^{k\times n!+1}$. One next takes $B = \{a \in B_{k,n} : (\forall l, t < m)(\forall i, j \in n)(R_l \sim R_t, \mathsf{s}_{[i,j]}R_l \leq a \implies \mathsf{s}_{[i,j]}R_t \leq a)\}$, then $G \subseteq B$, $R \in B$, and B is closed under the operations, so that $\mathfrak{A}_n \subseteq \mathfrak{B} \subseteq \mathsf{split}(\mathfrak{A}_n, R, m)$, where $\mathfrak{B}$ is the algebra with universe B. Furthermore, $\mathfrak{B}$ is the smallest such subalgebra of $\mathsf{split}(\mathfrak{A}_n, R, m)$, where for each $i, j < n$, $\mathsf{s}_{[i,j]}R$ is partitioned into $p < m$ many parts cylindrically equivalent to $\mathsf{s}_{[i,j]}R$. The non–representability of the algebra $\mathsf{split}(\mathfrak{A}_n, R, m)$ can be pinned down to the existence of 'one more extra atom' leading to the incomptability condition $|U_0| < m$ (= number of subatoms) witnessed by the term τ using diagonal elements. Using that $|\mathfrak{P}| = m$, we showed that a representation h of $\mathsf{split}(\mathfrak{A}_n, R, m)$ that respects R, has to respect the atoms below it, and this forces that $|U_0| \geq m$, which contradicts the construction of $\mathfrak{A}_n$. But this cannot happen with $\mathfrak{B}$, because $p < m$ (by the condition $|G| \leq k$)), so that this 'one more extra atom and possibly more' vanish in $\mathfrak{B}$. In representing $\mathfrak{B}$, we use the following *optimal compatibility condition* between the cardinality of $|U_0|$ and the number of *concrete* copies of R represented genuinely in $\mathfrak{B}$: **(*) If $\mathfrak{m}$ is any cardinal, α is an ordinal $\geq \omega$, $(U_i : i \in \alpha)$ is a system of sets each having cardinality $\geq \mathfrak{m}$, and $U \supseteq \bigcup\{U_i : i \in \alpha\}$, then there is a partition $(R_j : j < \mathfrak{m})$ of $R = \prod_{i\in I} U_i$ such that $\mathsf{c}_i^U R_j = \mathsf{c}_i^U R$ for all $i < \alpha$ and $j < \mathfrak{m}$ [1, Lemma 3].** Representing $\mathfrak{B}$ is done by embedding it into a representable algebra $\mathfrak{C}$ having the same top element as $\mathfrak{A}_n$, namely, ${}^{\omega}U$, where $R \in \mathfrak{C}$ is partitioned *concretely into* $m - 1$ *real atoms*, that is, there exists $R_l \subseteq {}^{\omega}U$, $l < m - 1$ *real atoms* in $\mathfrak{C}$ such that for all $i < j < n$, $\mathsf{S}_{[i,j]}R = \mathsf{S}_{[i,j]} \bigcup_{l<m-1} R_l = \bigcup_{l<m-1} \mathsf{S}_{[i,j]}R_l$ and $\mathsf{C}_i R_l = \mathsf{C}_i R$ for all $l < m - 1$ and $i < \omega$. *This concrete partition exists by (*) because* $|U_0| = m - 1$ and by the condition $|G| \leq k$, the value of p, which is the new number of subatoms of R in $\mathfrak{B}$ (depending on G) cannot exceed $m - 1$.

(3) Forming the directed union getting the required algebra: For fixed k, obtaining the algebras $\mathfrak{B}_{k,n} = \mathsf{split}(\mathfrak{A}_n, R, m)$ for each each $2 < n < \omega$ we proceed as follows. The constructed non–representable algebras form a chain in the following sense: For $2 < n_1 < n_2$, $\mathfrak{B}_{k,n_1}$ embeds into $\mathfrak{B}_{k,n_2}$, where the last algebra is the reduct obtained by discarding substitution operations not in the signature of the former, that is the substitution operations $\mathsf{s}_{[i,j]} : i, j \geq n_1, i \neq j$. Take the directed union $\mathfrak{B}_k = \bigcup_{n\in\omega\sim 3} \mathfrak{B}_{k,n}$ having the signature of QEA_ω. The cylindric reduct of $\mathfrak{B}_k$ is not representable because the cylindric reduct of every $\mathfrak{B}_{k,n}$ is not representable. Using that the k generated subalgebras of $\mathfrak{B}_{k,n}$ for each $2 < n < \omega$ are repesentable, it follows without difficulty that the k–generated subalgebras of $\mathfrak{B}_k$ remain representable.

One constructs such algebra $\mathfrak{B}_k$ having the signature of QEA_ω for each positive k. But one can even go further, by showing that the diagonal free reduct of $\mathfrak{B}_k$ so constructed is in RQA_ω for each k, by showing that this is the case for every $\mathfrak{B}_{k,n}(n > 2)$. Recall that for fixed positive k and $2 < n < \omega$, the algebra $\mathfrak{B}_{k,n}$ is not representable because of the incompatibity of $|U_0| <$ number of subatoms. One now adds 'one extra element or more' to $|U_0|$ forming W_0 to compensate for such an incompatibility. The diagonal free reduct of $\mathfrak{B}_{k,n}$ can now be represented by a set algebra $\mathfrak{C}$ obtained by splitting an ω–ary relation $R = W_0 \times \prod_{i\in\omega} U_i$ where $|W_0| \geq m + 1$ and (as before) $|U_i| = m$, $i \in \omega$, in a set algebra generated by R, into

m *real atoms*, as described in (*). Here, in the *absence of diagonal elements*, we cannot count the elements in $|W_0|$ (like we did with $|U_0|$ using the term τ defined above), so adding this element to U_0 does not clash with the concrete interpretation of the other operations. In short, $\mathfrak{Rd}_{qa}\mathfrak{B}_{k,n}$ can be represented via $\mathfrak{C}$. This gives the required relative non–finite axiomatizability result. $\square$

Corollary 1.1. *The variety* RQEA_ω *cannot be axiomatized by a set of universal formulas containing finitely many variables over* RSc_ω *nor over* RDf_ω*. Furthermore, the variety* RCA_ω *cannot be axiomatized by a set of universal formulas containing finitely many variables over* RSc_ω *nor over* RDf_ω*.*

Proof. For each positive k, using the notation in the previous proof, $\mathfrak{Rd}_{sc}\mathfrak{B}_k \in \mathsf{RSc}_\omega$ and $\mathfrak{Rd}_{df}\mathfrak{B}_k \in \mathsf{RDf}_\omega$, while $\mathfrak{B}_k \notin \mathsf{RQEA}_\omega$. This gives the required in the first item By observing that in fact, for each positive k, $\mathfrak{Rd}_{ca}\mathfrak{B}_k \notin \mathsf{RCA}_\omega$, we get the required in the second item reproving a result of Andreka's [1]. $\square$

2 Blow up and blur constructions

2.1 Blowing up and blurring a finite Maddux algebra

We elaborate on the construction on [3]. The atom structure of an atomic algebra $\mathfrak{A}$ will be denoted by $\mathsf{At}(\mathfrak{A})$ or simply $\mathsf{At}\mathfrak{A}$. A class L of Boolean algebra with operators is *atom canonical* if whenever $\mathfrak{A} \in \mathsf{L}$ is completey additive, then its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure (in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$) is also in L. For a relation atom structure α and $n > 2$, $\mathsf{Mat}_n(\alpha)$ denotes the set of all n by n basic matrices on α.
Let $\mathfrak{R}$ be a relation algebra, with non–identity atoms I and $2 < n < \omega$. Assume that $J \subseteq \wp(I)$ and $E \subseteq {}^3\omega$. (J, E) is an n*–blur* for $\mathfrak{R}$, if J is a *complex* n*–blur* and the tenary relation E is an *index blur* defined as in item (ii) of [3, Definition 3.1]. We say that (J, E) is a *strong* n*–blur*, if it (J, E) is an n–blur, such that the complex n–blur satisfies: $(\forall V_1, \ldots V_n, W_2, \ldots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\mathsf{safe}(V_i, W_i, T)$ (with notation as in [3]). The following theorem concisely summarizes the blow up and blur construction in [3] and says some more easy facts. We denote the relation algebra $\mathfrak{Bb}(\mathfrak{R}, J, E)$ with atom structure $\mathbf{At}$ obtained by blowing up and blurring $\mathfrak{R}$ (with underlying set is denoted by At on [3, p.73]) by $\mathsf{split}(\mathfrak{R}, J, E)$. By the same token, we denote the algebra $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$ as defined in [3, Top of p. 78] by $\mathsf{split}_l(\mathfrak{R}, J, E)$. This switch of notation is motivated by the fact that we wish to emphasize the role of *splitting some (possibly all) atoms into infinitely subatoms during blowing up and blurring a finite algebra.*

Theorem 2.1. *Let* $2 < n \leq l < m \leq \omega$*. Let* $\mathfrak{R}$ *be a finite relation algebra with an* l*–blur* (J, E) *where* J *is the* l*–complex blur and* E *is the index blur.*
(1) The set of l *by* l*–dimensional matrices* $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$ *is an* l*–dimensional cylindric basis, that is a weakly representable atom structure [3, Theorem 3.2]. The algebra* $\mathsf{split}_l(\mathfrak{R}, J, E)$ *with atom structure* $\mathbf{At}_{ra}$ *is in* RCA_l*. Furthermore,* $\mathfrak{R}$ *embeds into* $\mathfrak{Cm}\mathbf{At}$ *which embeds into* $\mathsf{Ra}\mathfrak{Cm}(\mathbf{At}_{ca})$*. If* (J, E) *is a strong* m*–blur for* $\mathfrak{R}$*, then* (J, E) *is a strong* l*–blur for* $\mathfrak{R}$*,* $\mathsf{split}_l(\mathfrak{R}, J, E) \cong \mathsf{Nr}_l\mathsf{split}_m(\mathfrak{R}, J, E)$ *and*

$\mathsf{split}(\mathfrak{R}, J, E) \cong \mathsf{Ra}(\mathsf{split}_l(\mathfrak{R}, J, E)) \cong \mathsf{Ra}(\mathsf{split}_m(\mathfrak{R}, J, E))$.
(2) For every $n < l$, there is an $\mathfrak{R}$ having a strong l–blur (J, E) but no infinite representations (representations on an infinite base). Hence the atom structures defined in the first item for this specific $\mathfrak{R}$ are not strongly representable.
(3) Let $m < \omega$. If $\mathfrak{R}$ is a finite relation algebra having a strong l–blur, and no m–dimensional hyperbasis, then $l < m$. If $n = l < m < \omega$ and $\mathfrak{R}$ as above has an n blur (J, E) and no infinite m–dimensional hyperbasis, then $\mathfrak{Cm}\mathsf{At}(\mathsf{split}(\mathfrak{R}, J, E))$ and $\mathfrak{Cm}\mathsf{At}(\mathsf{split}_l(\mathfrak{R}, J, E))$ are outside $\mathbf{S}\mathsf{RaCA}_m$ and $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, respectively, and the latter two varieties are not atom–canonical.

Proof. [3, Lemmata 3.2, 4.2, 4.3]. We start by an outline of (1). Let $\mathfrak{R}$ be as in the hypothesis. Let $3 < n \leq l$. We blow up and blur $\mathfrak{R}$. $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many. $\mathfrak{R}$ is blurred by using a finite set of blurs (or colours) J. This can be expressed by the product $\mathbf{At} = \omega \times \mathsf{At}\mathfrak{R} \times J$, which will define an infinite atom structure of a new relation algebra. Then two partitions are defined on $\mathbf{At}$, call them P_1 and P_2. Composition is re-defined on this new infinite atom structure; it is induced by the composition in $\mathfrak{R}$, and a ternary relation E on ω, that 'synchronizes' which three rectangles sitting on the i, j, k E–related rows compose like the original algebra $\mathfrak{R}$. The first partition P_1 is used to show that $\mathfrak{R}$ embeds in the complex algebra of this new atom structure, namely $\mathfrak{Cm}\mathbf{At}$, The second partition P_2 divides $\mathbf{At}$ into *finitely many (infinite) rectangles*, each with base $W \in J$, and the term algebra (denoted in [3] by $\mathfrak{Bb}(\mathfrak{R}, J, E)$) over $\mathbf{At}$, denoted here by $\mathsf{split}(\mathfrak{R}, J, E)$, consists of the sets that intersect co–finitely with every member of this partition. The algebra $\mathsf{split}(\mathfrak{R}, J, E)$ is representable using the finite number of blurs. Because (J, E) is a complex set of l–blurs, this atom structure has an l–dimensional cylindric basis, namely, $\mathbf{At}_{ca} = \mathsf{Mat}_l(\mathbf{At})$. The resulting l–dimensional cylindric term algebra $\mathfrak{Tm}\mathsf{Mat}_l(\mathbf{At})$, and an algebra $\mathfrak{C}$ having atom structure $\mathbf{At}_{ca}$ (denoted in [3] by $\mathfrak{Bb}_l(\mathfrak{R}, J, E)$) and denoted now by $\mathsf{split}_l(\mathfrak{R}, J, E)$ such that $\mathfrak{Tm}\mathsf{Mat}_l(\mathbf{At}) \subseteq \mathfrak{C} \subseteq \mathfrak{Cm}\mathsf{Mat}_l(\mathbf{At})$ is shown to be representable. Assume that the m–blur (J, E) is strong. Then by [3, item (3) pp. 80], $\mathsf{split}_l(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_l\mathsf{split}_m(\mathfrak{R}, J, E)$.
For (2): Like in [3, Lemma 5.1], one takes $l \geq 2n - 1$, $k \geq (2n - 1)l$, $k \in \omega$. The Maddux integral relation algebra $\mathfrak{E}_k(2, 3)$ where k is the number of non-identity atoms is the required $\mathfrak{R}$. In this algebra a triple (a, b, c) of non–identity atoms is consistent $\iff |\{a, b, c\}| \neq 1$, i.e only monochromatic triangles are forbidden.
We prove (3). Let (J, E) be the strong l–blur of $\mathfrak{R}$. Assume for contradiction that $m \leq l$. Then we get by [3, item (3), p.80], that $\mathfrak{A} = \mathsf{split}_n(\mathfrak{R}, J, E) \cong \mathfrak{Nr}_n\mathsf{split}_l(\mathfrak{R}, J, E)$. But the cylindric l–dimensional algebra $\mathsf{split}_l(\mathfrak{R}, J, E)$ is atomic, having atom structure $\mathsf{Mat}_l\mathsf{At}(\mathsf{split}(\mathfrak{R}, J, E))$, so $\mathfrak{A}$ has an atomic l–dilation. So $\mathfrak{A} = \mathsf{Nr}_n\mathfrak{D}$ where $\mathfrak{D} \in \mathsf{CA}_l$ is atomic. But $\mathfrak{R} \subseteq_c \mathsf{RaNr}_n\mathfrak{D} \subseteq_c \mathsf{Ra}\mathfrak{D}$. By [8, Theorem 13.45 (6) $\iff$ (9)], $\mathfrak{R}$ has a complete l–flat representation, thus it has a complete m–flat representation, because $m < l$ and $l \in \omega$. This is a contradiction. For the second part. Let $\mathfrak{B} = \mathsf{split}_n(\mathfrak{R}, J, E)$. Then, since (J, E) is an n blur, $\mathfrak{B} \in \mathsf{RCA}_n$. But $\mathfrak{C} = \mathfrak{Cm}\mathsf{At}\mathfrak{B} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$, because $\mathfrak{R} \notin S\mathsf{RaCA}_m$, $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which, in turn, embeds into $\mathsf{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{B}$. Similarly, $\mathsf{split}(\mathfrak{R}, J, E) \in \mathsf{RRA}$ and $\mathfrak{Cm}(\mathsf{At}\mathsf{split}(\mathfrak{R}, J, E)) \notin \mathbf{S}\mathsf{RaCA}_m$. Non atom –canonicity follows. $\square$

2.2 Blowing up and blurring finite rainbow algebras

In theorem 2.1, we used a single blow up and blur construction to prove non-atom–canonicity of RRA and $\mathsf{RCA_n}$ for $2 < n < \omega$. This constructon is based on relation algebras that have an n–dimensional cylindric basis denoted above by $\mathfrak{E}_k(2,3)$. To obtain finer results, we use *two blow up and blur constructions*. For the RA case we blow up and blur the finite rainbow relation algebra (denoted below by) $\mathbf{R}_{4,3}$ and for the CA case we blow up and blur the finite rainbow CA_n (denoted below by) $\mathfrak{A}_{n+1,n}$. While the Maddux algebra used in [3] and in the second item of theorem 2.1, $\mathfrak{E}_k(2,3)$ has an n—dimensional cylindric basis (by suitably choosing k), the relation rainbow algebra $\mathbf{R}_{4,3}$ does not have a 4–dimensional cylindric basis. So for CAs we start anew.

Relation algebras: We briefly review the blow up and blur construction in [8, 17.32, 17.34, 17.36] for relation algebras. Let $2 \leq n \leq \omega$ and $r \leq \omega$. Let $\mathfrak{R}$ be an atomic relation algebra. Then the r–rounded game $G^n_r(\mathsf{At}\mathfrak{R})$ [8, Definition 12.24] is the (usual) atomic game played on networks of an atomic relation algebra $\mathfrak{R}$ using n nodes.

Let L be a relational signature. Let G (the greens) and R (the reds) be L structures and $p, r \leq \omega$. The game $\mathsf{EF}^p_r(\mathsf{G},\mathsf{R})$, defined in [8, Definition 16.1.2], is an Ehrenfeucht–Fraïssé forth 'pebble game' with r rounds and p pairs of pebbles. In [8, 16.2], a relation algebra *rainbow atom structure* is associated for relational structures G and R. We denote by $\mathbf{R}_{A,B}$ the (full) complex algebra over this atom structure. The **Rainbow Theorem** [8, Theorem 16.5] states that: *If* G,R *are relational structures and* $p, r \leq \omega$, *then* $\exists$ *has a winning strategy in* $G^{2+p}_{1+r}(\mathbf{R}_{\mathsf{G},\mathfrak{R}}) \iff$ *she has a winning strategy in* $\mathsf{EF}^p_r(\mathsf{G},\mathsf{R})$.

For $5 \leq l < \omega$, RA_l is the class of relation algebras whose canonical extensions have an l–dimensional relational basis [8, Definition 12.30]. RA_l is a variety containing properly the variety $\mathbf{S}\mathsf{RaCA}_l$. Furthermore, $\mathfrak{R} \in \mathsf{RA}_l \iff \exists$ has a winning strategy in $G^n_\omega(\mathsf{At}\mathfrak{R})$. Cf. [8, Proposition 12.31] and [8, Remark 15.13]. We now show:

Theorem 2.2. *For any* $k \geq 6$, *the varieties* RA_k *and* $\mathbf{S}\mathsf{RaCA}_k$ *are not atom–canonical.*

Proof. We follow the notation in [9, lemmas 17.32, 17.34, 17.35, 17.36] with the sole exception that we denote by m (instead of $\mathbf{K}_m$) the complete irreflexive graph on m defined the obvious way; that is we identify this graph with its set of vertices. Fix $2 < n < m < \omega$. Let $\mathfrak{R} = \mathbf{R}_{m,n}$. Then by the rainbow theorem $\exists$ has a winning strategy in $G^{m+2}_{m+1}(\mathsf{At}\mathfrak{R})$, since it clealy has a winning strategy in the Ehrenfeucht–Fraïssé game $\mathsf{EF}^m_m(m,n)$ because m is 'longer' than n. Then $\mathfrak{R} \notin \mathsf{RA}_{m+2}$ by [8, Propsition 12.25, Theorem 13.46 (4) $\iff$ (5)], so $\mathfrak{R} \notin \mathbf{S}\mathsf{RaCA}_{m+2}$. Next one 'splits' every red atom to ω–many copies obtaining the infinite atomic countable (term) relation algebra denoted in *op.cit* by $\mathcal{T}$, which we denote by $\mathsf{split}(\mathfrak{R},\mathsf{r},\omega)$ (blowing up the reds by splitting each into ω–many subatoms) with atom structure α, cf. [8, item (4) top of p. 532]. Then $\mathfrak{Cm}\alpha \notin \mathbf{S}\mathsf{RaCA}_{m+2}$ because $\mathfrak{R}$ embeds into $\mathfrak{Cm}\alpha$ by mapping every red to the join of its copies, and $\mathbf{S}\mathsf{RaCA}_{m+2}$ is closed under $\mathbf{S}$. Finally, one (completely) represents (the canonical extension of) $\mathsf{split}(\mathfrak{R},\mathsf{r},\omega)$ like in [8]. By taking $m = 4$ and $n = 3$ the required follows. □

Cylindric algebras: From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2. For an atomic $\mathfrak{A} \in \mathsf{CA}_n$, the ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the usual atomic ω–rounded game $G^m_\omega(\mathsf{At}\mathfrak{A})$ using m nodes, except that $\forall$ has the option to re–use the m nodes in play. We need the following lemma:

Lemma 2.3. *Let $2 < n < m$. Let K be any any class having signature between Sc and QEA, $\mathfrak{A} \in \mathsf{K}_n$ and $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{K}_m$, then $\exists$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$.*

For rainbow constructions for CAs, we follow [7, 9]. Fix $2 < n < \omega$. Given relational structures G (the greens) and R (the reds) the rainbow atom structure of a QEA_n consists of equivalence classes of surjective maps $a : n \to \Delta$, where Δ is a coloured graph. A coloured graph is a complete graph labelled by the rainbow colours, the greens $\mathsf{g} \in \mathsf{G}$, reds $\mathsf{r} \in \mathsf{R}$, and whites; and some $n-1$ tuples are labelled by 'shades of yellow'. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. A red triple $(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*})$ $i, j, j', k', i^*, k^* \in \mathsf{R}$ is not allowed, unless $i = i^*$, $j = j'$ and $k' = k^*$, in which case we say that the red indices match, cf.[7, 4.3.3]. The equivalence relation relates two such maps $\iff$ they essentially define the same graph [7, 4.3.4]. We let $[a]$ denote the equivalence class containing a. The accessibilty (binary relations) corresponding to cylindric operations are like in [7]. For transpositions $([i, j],\ i < j < n)$ they are defined as follows: $[a]S_{[i,j]}[b] \iff a = b \circ [i, j]$. For $2 < n < \omega$, we use the graph version of the games $G^m_\omega(\beta)$ and $\mathbf{G}^m(\beta)$ where β is a QEA_n rainbow atom structure, cf. [7, 4.3.3]. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context. We let $\mathfrak{Rd}_{df}$ denotes 'diagonal free reduct' and $\mathfrak{Rd}_{sc}$ denote 'Sc reduct'.

Theorem 2.4. *Let n be a finite ordinal > 2 and K is a class between Sc and QEA. Assume that $m \geq n + 3$. Then the varieties $\mathbf{S}\mathsf{Nr}_n\mathsf{K}_m$ and RDf_n, are not atom–canonical.*

Proof. The idea for CAs is like that for RAs by blowing up and blurring (the CA reduct of) $\mathfrak{A}_{n+1,n}$ in place of $\mathbf{R}_{4,3}$. We work with $m = n + 3$ and any K between Sc and QEA. This gives the result for any larger m. Fix $2 < n < \omega$.
Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure At: Take the finite quasi–polyadic equality algebra rainbow algebra $\mathfrak{A}_{n+1,n}$ where the reds R is the complete irreflexive graph n, and the greens are $\mathsf{G} = \{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}^i_0 : 1 \leq i \leq n+1\}$, endowed with the polyadic operations. Denote its finite atom structure by $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{A}_{n+1,n})$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure $\mathbf{At}$. The resulting atom structure after 'splitting the reds', namely, $\mathbf{At}$, is like the weakly (but not strongly) representable atom structure of the atomic, countable and simple algebra $\mathfrak{A}$ as defined in [10, Definition 4.1]; the sole difference is that we have $n+1$ greens and not ω–many as is the case in [10]. We denote the algebra $\mathfrak{Tm}\mathbf{At}$ by $\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ short hand for blowing up $\mathfrak{A}_{n+1,n}$ by splitting each *red graphs (atoms)* into ω many. By a red graph is meant (an equivalence class of) a surjection $a : n \to \Delta$, where Δ is a coloured graph in the rainbow signature

of $\mathfrak{A}_{n+1,n}$ with at least one edge labelled by a red label (some r_{ij}, $i<j<n$). It can be shown exactly like in [10] that $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using a shade of red ρ *outside* the rainbow signature, when she is forced a red; [10, Proposition 2.6, Lemma 2.7]. Using this, one proves like in *op.cit* that $\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ is representable as a set algebra having top element ${}^n\mathsf{M}$. (The term algebra in [10]; which is the subalgebra generated by the atoms of $\mathfrak{A}$ as defined in [10, Definition 4.1] is just $\mathsf{split}(\mathfrak{A}_{\omega,n}, \mathsf{r}, \omega)$.)
Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathsf{At}(\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)))$: Let CRG_f be the class of coloured graphs on $\mathbf{At_f}$ and CRG be the class of coloured graph on $\mathbf{At}$. We can assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_a \sim N_b$, $(M, N \in \mathsf{CGR})$ $\iff$ they are identical everywhere except at possibly at red edges:
$M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$.
We say that M_a is a *copy of* N_b if $M_a \sim N_b$ (by symmetry N_b is a copy of M_a.) Indeed, the relation 'copy of' is an equivalence relation on $\mathbf{At}$. An atom M_a is called a *red atom*, if M has at least one red edge. Any red atom has ω many copies that are *cylindrically equivalent*, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\mathsf{nodes}(N) = \mathsf{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{Cm}\mathbf{At}$, we write M_a for $\{M_a\}$ and we denote suprema taken in $\mathfrak{Cm}\mathbf{At}$, possibly finite, by $\sum$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At_f}$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathsf{At_f}$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. It can be checked that Θ is an injective a homomorphism, hence Θ is the required embedding.
$\forall$ has a winning strategy in $G^{n+3}\mathsf{At}(\mathfrak{A}_{n+1,n})$: It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1$ and n in $n+1$ rounds, namely, the game $\mathsf{EF}_{n+1}^{n+1}(n+1, n)$ [9, Definition 16.2]. $\forall$ lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game, to the graph game on $\mathbf{At}_f = \mathsf{At}(\mathfrak{A}_{n+1,n})$ [7, pp. 841] forcing a win using $n+3$ nodes. He bombards $\exists$ with cones [1] having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). By lemma 2.3, $\mathfrak{Rd}_{sc}\mathfrak{A}_{n+1,n} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{Sc}_{n+3}$. Since $\mathfrak{A}_{n+1,n}$ is finite, then $\mathfrak{Rd}_{sc}\mathfrak{A}_{n+1,n}$ is not in $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{n+3}$. But $\mathfrak{A}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Rd}_{sc}\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathfrak{Rd}_{sc}\mathsf{At}\mathfrak{A}$ is outside $\mathbf{S}\mathsf{Nr}_n\mathsf{Sc}_{n+3}$, too. Since $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is generated using infinite (countable) unions by $\{x \in \mathfrak{C} : \Delta x \neq n\}$, then easily adapting the proof of [4, Lemma 5.1.50, Theorem 5.1.51] and [4, Theorem 5.4.26], we get that $\mathfrak{Rd}_{df}\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RDf}_n$. This proves the non–atom canonicity of RDf_n, too. □

Hodkinson uses an 'overkill' of infinitely many greens showing non–representability of $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathsf{At}(\mathfrak{A}_{\omega,n}, \mathsf{r}, \omega)$ with $\mathfrak{A}$ as defined in [10, Definition 4.1]. From [10], we

[1] Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M *an i - cone* if $M(x_0, z) = \mathsf{g}_0^i$ and for every $1 \leq j \leq m-2$, $M(x_j, z) = \mathsf{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the **base of the cone**, z the **apex of the cone** and i the **tint of the cone**.

know that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ for some $m > n$, but the (semantical) argument used in *op.cit* does not give any information on the value of such m. By truncating the greens to be $n+1$, and using instead a syntactical blow up and blur construction, we could pin down such a value of m, namely, $m = n+3$ (=number of greens +2) by showing that $\mathfrak{Cm}\mathsf{At}(\mathsf{split}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)) \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$.

3 Non–elementary classes

Cylindric algebras: For a class L, we write $\mathbf{El}\mathsf{L}$ for the elementary closure of L. We write $\mathsf{At}\mathsf{L}$ for the class $\{\mathsf{At}\mathfrak{A} : \mathfrak{A} \in \mathbf{K} \cap \mathbf{At}\}$ (of first order structures). We now prove:

Theorem 3.1. *Let $2 < n < \omega$ and let $k \geq 3$. For any class $\mathbf{K}$, such that $\mathsf{CRCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary. In particular, any class between $\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$ and $\mathsf{At}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$ is not elementary.*

Proof. We start with proving: (*) Any class $\mathbf{K}$ between $\mathsf{CRCA}_n \cap \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ is not elementary: Take the rainbow–like CA_n, call it $\mathfrak{C}$, based on the ordered structure $\mathbb{Z}$ and $\mathbb{N}$. The reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega(= \mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathbb{N}$ specified above, but now the triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$.
(1) $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$: This can be proved using lemma 2.3 by showing that $\forall$ has a winning strategy in $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C})$. His winning strategy is to bombard $\exists$ with cones having common base and distinct green tints. $\exists$ has to label edges between appexes of cones created during the game by reds. The newly added consistency condition of 'order preserving' restricts $\exists$'s 'red choices'. To conform to the rules of the play, $\exists$ is forced to play reds r_{ij} with the first index forming a decreasing sequence in $\mathbb{N}$. Having the option to re–use the nodes in play, $\forall$ needs to use and reuse *exactly* $n+3$ nodes to force a win in ω rounds (but not before).
(2) $\mathfrak{C} \equiv \mathfrak{B}'$ for some countable $\mathfrak{B}' \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$: One can define a k–rounded game $\mathbf{H}_k$ for $k \leq \omega$, played on so–called λ*–neat hypernetworks* on an atom structure. This game besides the standard cylindrifier move (modified to λ–neat hypernetworks), offers $\forall$ two new amalgamation moves. (We omit the highly technical definitions). One shows that $\exists$ has a winning strategy in $\mathbf{H}_k(\mathsf{At}\mathfrak{C})$ for all $k < \omega$, hence using ultrapowers followed by an elementary chain argument, $\exists$ has a winning strategy in $\mathbf{H}_\omega(\alpha)$ for a countable atom structure α, such that $\mathsf{At}\mathfrak{C} \equiv \alpha$. The game $\mathbf{H}$ is designed so that the winning strategy of $\exists$ in $\mathbf{H}_\omega(\alpha)$ implies that $\alpha \in \mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$ and that $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Let $\mathfrak{B}' = \mathfrak{Tm}\alpha$. Then $\mathfrak{B}' \subseteq_d \mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA_n} \subseteq \mathbf{K}$, $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3} \supseteq \mathbf{K}$, and $\mathfrak{C} \equiv \mathfrak{B}'$ proving (*). Observe that this already proves the second part, because the atom structure of an algebra is first order definable in the algebra, so that $\alpha \equiv \mathsf{At}\mathfrak{C}$. Also $\alpha \in \mathsf{At}\mathsf{Nr}_n\mathsf{CA}_\omega$. Finally, $\mathsf{At}\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$ because for any $m > n$, and any atomic $\mathfrak{D} \in \mathsf{CA}_n$, $\mathfrak{D} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m \iff \mathsf{At}\mathfrak{D} \in \mathsf{At}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$.
To complete the proof, we first need to slighty modify the construction in [12, Lemma

5.1.3, Theorem 5.1.4] reformulating it as a 'splitting argument'. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ constructed in *op.cit* satisfy that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, $\mathfrak{A}$ and $\mathfrak{B}$ are not atomic, but they it can be fixed that they are to be so giving the same result, by interpreting the uncountably many tenary relations in the signature of M defined in [12, Lemma 5.1.3], which is the base of $\mathfrak{A}$ and $\mathfrak{B}$ to be *disjoint* in M, not just distinct. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in *op.cit* lift verbatim to any such n. Let $u \in {}^nn$. Write $\mathbf{1}_u$ for χ_u^{M} (denoted by 1_u (for $n = 3$) in [12, Theorem 5.1.4].) We denote by $\mathfrak{A}_u$ the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathbf{1}_u\}$ and similarly for $\mathfrak{B}$, writing $\mathfrak{B}_u$ short hand for the Boolean algebra $\mathfrak{Rl}_{\mathbf{1}_u}\mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathbf{1}_u\}$. It can be shown that $\mathfrak{A} \equiv_\infty \mathfrak{B}$. Using that M has quantifier elimination we get, using the same argument in *op.cit* that $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. The property that $\mathfrak{B} \notin \mathsf{Nr}_n\mathsf{CA}_{n+1}$ is also still maintained. To see why consider the substitution operator ${}_n\mathsf{s}(0,1)$ (using one spare dimension) as defined in the proof of [12, Theorem 5.1.4]. Assume for contradiction that $\mathfrak{B} = \mathsf{Nr}_n\mathfrak{C}$, with $\mathfrak{C} \in \mathsf{CA}_{n+1}$. Let $u = (1, 0, 2, \ldots n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term ${}_n\mathsf{s}(0,1)$ acts like a substitution operator corresponding to the transposition $[0,1]$; it 'swaps' the first two co–ordinates. Now one can show that ${}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u \subseteq \mathfrak{B}_{[0,1]\circ u} = \mathfrak{B}_{Id}$, so $|{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u|$ is countable because $\mathfrak{B}_{Id}$ was forced by construction to be countable. But ${}_n\mathsf{s}(0,1)$ is a Boolean automorpism with inverse ${}_n\mathsf{s}(1,0)$, so that $|\mathfrak{B}_u| = |{}_n\mathsf{s}(0,1)^{\mathfrak{C}}\mathfrak{B}_u| > \omega$, contradiction.

We show that $\mathfrak{B}$ is in fact outside $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$ getting the required: Take κ the signature of M to be 2^{2^ω} and assume for contradiction that $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathbf{At}$. Then $\mathfrak{B} \subseteq_d \mathsf{Nr}_n\mathfrak{D}$, for some $\mathfrak{D} \in \mathsf{CA}_\omega$ and $\mathsf{Nr}_n\mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathsf{Nr}_n\mathfrak{D}$. Then $\mathfrak{Rl}_{Id}\mathfrak{B} \subseteq_d \mathfrak{Rl}_{Id}\mathfrak{C}$. Since $\mathfrak{C}$ is atomic, then $\mathfrak{Rl}_{Id}\mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|\mathfrak{Rl}_{Id}\mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} = \mathsf{Nr}_n\mathsf{CA}_\omega$.) By the choice of κ, we get that $|\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. By density, $\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C} \subseteq \mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{B}$, so $|\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{B}| \geq |\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{C}| > \omega$. But by the construction of $\mathfrak{B}$, we have $|\mathfrak{Rl}_{Id}\mathfrak{B}| = |\mathsf{At}\mathfrak{Rl}_{Id}\mathfrak{B}| = \omega$, which is a contradiction and we are done. □

Relation algebras: For an ordinal α, let $\mathfrak{R}^\alpha$ be the relation algebra defined in [11]. $\mathfrak{R}^\alpha$ is obtained from $\mathfrak{R}$ defined in *op.cit* by splitting the red atom $\mathsf{r}(0)$ into α many parts. Let $\mathfrak{n} \geq 2^{\aleph_0}$. Then $\mathfrak{R}^{\mathfrak{n}} \equiv_{\infty,\omega} \mathfrak{R}^\omega$, $\mathfrak{R}^\omega \in \mathsf{RaCA}_\omega$ and $\mathfrak{R}^{\mathfrak{n}} \notin \mathsf{RaCA}_5$ and $\mathfrak{R}^{\mathfrak{n}}$ has no complete representation. We readily conclude:

Theorem 3.2. *Any class between* RaCA_ω *and* RaCA_5*, as well as the class of completely representable* RA*s, are not closed under* $\equiv_{\infty,\omega}$ *[11].*

The last part of theorem 2.3 is an improvement of the result in [7] on RAs. The following is also an improvement of the result in [6] (by [5, Theorem 36]):

Theorem 3.3. *Any class between* $\mathbf{S}_d\mathsf{RaCA}_\omega \cap \mathsf{CRRA}$ *and* $\mathbf{S}_c\mathsf{RaCA}_5$ *and any class between* AtRaCA_ω *and* $\mathsf{At}\mathbf{S}_c\mathsf{RaCA}_5$ *are not elementary.*

Proof. One uses the arguments in [5, Theorem 39, 45], but resorting to the game H_k $(k < \omega)$, as defined for relation algebras [5, Definition 28]. Now we have the countable relation algebra atom structure β based on $\mathbb{N}$ and $\mathbb{Z}$ as defined in [5], for which $\exists$ has a winning strategy in $H_k(\mathfrak{Cm}\beta)$, for all $k < \omega$, and $\forall$ has a winning strategy in $F^5(\beta)$ with F^5 as in [5, Definition 28]. By the RA analogue of lemma

2.3 proved in [5], we get that $\mathfrak{Cm}\beta \notin \mathbf{S}_c\mathsf{RaCA}_5$. The usual argument of taking an ultrapower of $\mathfrak{Cm}\beta$, followed by a downward elementary chain argument, one gets a countable atom structure α, such that $\mathfrak{Tm}\alpha \equiv \mathfrak{Cm}\beta$ and $\exists$ has a winning strategy in $H(\alpha)$, so using exactly the same argument in [5] allowing infinite conjunction in [5, Theorem 39], we get $\mathfrak{Tm}\alpha \in \mathbf{S}_d\mathsf{RaCA}_\omega$. □

Application: *Clique guarded semantics* for CA_ns can be defined similarly to relation algebras. We consider (the locally well–behaved) m–square [8, Definition 13.4] and m–flat representations of $\mathfrak{A} \in \mathsf{CA}_n$ with $2 < n < m \leq \omega$ [8, Chapter 13]. L_n denotes n variable first order logic. Fix $2 < n \leq l < m \leq \omega$. Consider the statement $\Psi(l,m)$: *There is an atomic, countable and complete L_n theory T, such that the type Γ consisting of co–atoms is realizable in every m– square model, but any formula isolating this type has to contain more than l variables.* By an m–square model M of T we understand an m–square representation of the algebra $\mathfrak{Fm}_T$ with base M.
Let $\mathsf{VT}(l,m)) = \neg\Psi(l,m)$, short for **Vaught's theorem holds 'at the parameters l and m'** where by definition, we stipulate that $\mathsf{VT}(\omega,\omega)$ is just Vaught's theorem for $L_{\omega,\omega}$: Countable atomic theories have countable atomic models. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, it is likely and plausible that (**) $\mathsf{VT}(l,m) \iff l = m = \omega$. In other words: **Vaught's theorem holds only in the limiting case when $l \to \infty$ and $m = \omega$ and not 'before'. This was proved on the 'paths' (l,ω), $n \leq l < \omega$ (x axis) and $(n, n+k)$, $k \geq n+3$ (y axis) using two different blow up and blur constructions, given in theorems 2.1, 2.4, respectively.** In the next theorem, we put some pieces together. $\Psi(l,m)_f$ is the formula obtained from $\Psi(l,m)$ be replacing square by flat.

Theorem 3.4. *Let $2 < n \leq l < m \leq \omega$. Then we have the following list of implications: There exists a finite relation algebra $\mathfrak{R}$ algebra with a strong l–blur and no infinite m–dimensional hyperbasis $\implies$ there is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{CA}_n$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m–flat representation $\implies$ there is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{SNr}_n\mathsf{CA}_m$ $\implies$ there is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{A}$ has no complete m–flat representation $\implies$ there is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{A} \notin \mathbf{S}_c\mathsf{Nr}_n(\mathsf{CA}_m \cap \mathbf{At})$ $\implies$ $\Psi(l,m)_f$ is true $\implies$ $\Psi(l',m')_f$ is true for any $l' \leq l$ and $m' \geq m$.*
The same implications hold upon replacing infinite m–dimensional hyperbasis by m–dimensional basis and m–flat by m–square.

Proof. First $\implies$: From theorem 2.1. Second $\implies$: Due to the equivalence of existence of m–dilations and m– flat representations for an atomic $\mathfrak{A} \in \mathsf{CA}_n$. Third $\implies$: A complete m–flat representation of (any) $\mathfrak{B} \in \mathsf{CA}_n$ induces an m- flat representation of $\mathfrak{Cm}\mathsf{At}\mathfrak{B}$. Fourth $\implies$: Like second implication dealing now with $\mathbf{S}_c$ and *complete* m–flat represenations. Fifth $\implies$: By [4, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–flat model, for if M is an m–flat model omitting Γ, then M would be the base of a complete m–flat representation of $\mathfrak{A}$. But $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$, so using exactly the same (terminology and) argument in [3, Theorem 3.1] we get that any witness isolating Γ needs more than l–variables. Last $\implies$ follows from the definitions. $\square$

If for $m > n > 2$, there is a finite relation algebra $\mathfrak{R}_m$ having a strong $m-1$–blur (J, E) and no m–square representation, then $\Psi(m-1, m)$ is true. Indeed, the algebra $\mathfrak{C} = \mathsf{split}_n(R_m, J, E)$ will be in $\mathsf{Nr}_n\mathsf{CA}_{m-1} \cap \mathsf{RCA}_n$ but it will have no *complete* m–square representation. If this $\mathfrak{R}_m$ exists for every $2 < n < m < \omega$, then (**) would be true by the the 'square version' of the last implication in theorem 3.4.

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