Relativistic Spacetime from Events

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This talk has three main aims. This first aim is to present a remarkable connection between the Kantian-Aristotelian theory of the temporal continuum, as we find it in the Critique of Pure Reason [4] and in the Physics [1], and work on the construction of time and spacetime from event structures, along the lines of A.G. Walker [10,11] and S. K. Thomason [8,9]. The second aim is that of outlining a formalization of Kant's theory of time which we have developed on the basis of the Walker-Thomason construction; we show that the unit interval $[0,1]$ is homeomorphic to the space of boundaries on the inverse limit of a directed diagram of event structures that are defined axiomatically, and that a particular event structure obtained from $[0,1]$ provides a universal model for the axioms. The third aim is that of relating this approach to recent work on finitary approximations of compact Hausdorff spaces, developed independently in Physics [7] and digital topology [5].

In [11] the late physicist A. G. Walker developed an axiomatic foundation for Milne's kinematical relativity in terms of point-like instants, which form a ground set $\mathcal{I}$, and a set of particles $\mathcal{R} \in \mathcal{P}(\mathcal{I})$. Walker formulates on this basis various axioms involving a relation of temporal order $< \subseteq \mathcal{I} \times \mathcal{I}$ on instants and a relation of signal correspondence $\wedge \subseteq \mathcal{R} \times \mathcal{R}$ on particles. It is not assumed at the outset, but it is a consequence of the axioms, that the instants in a particle $R \in \mathcal{R}$ are linearly ordered, and thus a particle is actually just its world-line. Walker then recovers, as models of his axioms, Milne's Kinematical relativity, and shows that his axiom system is consistent with respect to spacetimes in GR of zero or negative curvature. An axiom system for special relativity along Walker's lines, but with only the temporal relation of order as primitive, is given in [6].

In [11] Walker remarks the following:

> It may be argued that we are not in agreement with experience in taking our undefined element of time to be an instant, and that this element should be a *duration*, to be pictures as an interval.

> This is certainly true, and we hope later to replace the instants, temporal relations and the temporal axiom of the present paper by still more fundamental ideas in closer agreement with experience. These will give rise to instants as defined elements, and, except for signal correspondences which will then refer to durations, the remainder of the present paper will be valid. ([11], p.321)

Walker's philosophical concern was thus to provide a foundation for relativity theory in terms of a vocabulary and a set of fundamental notions having a clear phenomenological interpretation. A definition of instants in terms of durations, which can be thought of as extended subpaths of the world-line of a particle, is given in [10], a copy of which, however, I have been unable to obtain.

Nevertheless, Walker's construction of instants from events is presented formally in [8, 9] by logician S. K. Thomason. The emphasis lies here not on relativity theory, but on giving a mathematical foundation to the temporal continuum of ordinary experience, i.e., a mathematical theory of how human beings, which can only process finite amounts of information, come to think of time as an infinite linear continuum. To this effect, Thomason introduces a category of event structures, where an event structure is just a tuple $\mathcal{W} = (W, P, B, E)$ where W is a non-empty set of durations, and P, B, E are binary relations on W which are intuitively interpreted as "wholly precedes", "begins before" and "ends before". These relations are required to obey various intuitive axioms, among which one that ensures linearity of events. Morphisms between event structures are, as usual, just maps preserving the relations. An instant in an event structure $\mathcal{W}$ is then defined as a partition $(Past, Pres, Fut)$ of W into past, present, and future, which is required to satisfy various axioms expressed in terms of the primitive relations. The basic intuition here is that an instant in an event structure is just an "approximation" of a point-like instant, which arises wherever events can be separated, in agreement with the motto that "time is nature's way to keep everything from happening at once". It can be shown that instants arising from an event structure can be linearly ordered by letting $(Past, Pres, Fut) \subseteq (Past', Pres', Fut')$ if $Past \subseteq Past'$, i.e., inclusion of pasts suffices to linearly order instants.

Thomason then introduces a functor B from the category of event structures to the category of linear orders and order-preserving multivalued maps, which associates to each event structure $\mathcal{W}$ its linear order of instants $B(\mathcal{W})$ according to Walker's construction. A functor E from the category of linear order of instants to the category of event structures associates to any linear order of instants L the event structure generated by the linear order, taking as events the *formal* open intervals $\{(a, b) \in L \times L \mid a < b\}$. Various properties of these functors are then studied, and a natural transformation between the identity functor on the category of event structures and $E \circ B$ is introduced.

It is of interest that Thomason's and Walker's efforts in developing a mathematical theory of the temporal (Thomason) and spatio-temporal (Walker) continuum belong to a tradition of thought on the problem of the continuum which, following Feferman, can be termed "phenomenological" [2], and which goes back at least to Aristotle. Indeed, Aristotle held that the continuum is not composed of points but of extended parts, and that points are just boundaries or "places of limitation" of such parts. Similarly, he argues that time is not composed of instants but of events, and instants or "nows" are just "the links of time", i.e., they arise only as separations of events into past, present and future. Immanuel Kant adopted much of Aristotle's view on the temporal continuum, but he also departs from Aristotle in stating that the instant or "now" is not point-like, but has a certain breadth or extension, similarly to William James' notion of the "specious present" in the principles of psychology [3]. Furthermore, both Kant and Aristotle hold that every duration can be divided into extended durations to infinity, but that this infinity is only potential and is never actual, i.e., completed; hence points can only be introduced as limitations at some finite stage of division of time. In the last century we see this conception of the continuum resurface in the work of Weyl, who attempted to develop a mathematical theory of phenomenological time in order to provide a grounding in experience to the concept of the continuum. A clear statement of Weyl's programme is the following:

> Our examination of the continuum problem contributes to critical epistemology's investigation into the relations between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics (See [2])

It is clear from this quote that Weyl's concern is the same as Walker's and Thomason's, namely, to provide a formal theory of the temporal and spatiotemporal continuum on a purely phenomenological basis.

In light of the aforementioned similarities between Kant's theory of time and the Thomason-Walker construction of time from events, we recently developed a formalization of Kant's theory of time drawing from these ideas. Kant, however, ascribes various properties to time which are essentially topological, such as the propery of being "connected" in a much stronger way than the indecompasability of classical continuum, or that of being "continuous". We therefore modified Thomason's treatment and defined an event structure to be a tuple $\mathcal{W} = (W, E, \breve{E}, B, \breve{B}, O, \breve{O})$ where W is a set of events, and $E, \breve{E}, B, \breve{B}, O, \breve{O}$ are relations intuitively interpreted as "ends before", "does not end before", "begins after", "does not begin after", "overlaps", "does not overlap". The use of antonyms such as $\breve{E}, \breve{B}, \breve{O}$ instead of negations is important, since it allows one to express all the axioms on event structures as geometric implications, which are preserved by inverse limits. The category of event structures now has these tuples as objects, but the maps are required to preserve only $\breve{E}, \breve{B}, O$. Since $\breve{B}, \breve{E}$ are preorders, they give rise to two Alexandroff topologies on the set of events W. Thus one can equivalently see an event structure as a set W equipped with two Alexandroff topologies, a past and a future topology, and a proximity relation on $\mathcal{P}(W)$ given by O; maps between event structures become then proximally bi-continuous maps.

One can then define a notion of Walker's instant or boundary in topological terms, and prove that these boundaries are linearly ordered by the pasts and can moreover be constructed as fixpoints of a particular operation on subsets of events. We then introduced, following Thomason, a functor B from the category of event structures to the category of linear orders and multivalued maps, associating to every event structure its linear order of boundaries. We then showed that the unit interval $[0, 1]$ is homeomorphic to the space of boundaries $B(\mathcal{W})$ on the inverse limit $\mathcal{W}$ of a directed diagram $D = \{\mathcal{W}_s, f_{st}, \mathcal{I}\}$ of finite event structures and *retraction* maps, when equipped with the order topology. Furthermore, it can be shown that the unit interval can be equivalently recovered as the inverse limit of the $B(\mathcal{W}_s)$. We also showed that the unit interval $[0, 1]$ gives rise to an event structure Ω by considering the set of all intervals with rational endpoints as events, and interpreting the primitives in the obvious way in terms of the endpoints, and proved that this event structure provides a universal model for the axioms.

The method by means of which we recovered the unit interval as an inverse limit of finite event structures bears a very close resemblance to an approach to the approximation of arbitrary compact Hausdorff spaces by means of finite T_0 spaces which has been developed independently in Physics [7] and in digital topology [5]. In particular, it is shown in [5] that any compact Hausdorff space X is the Hausdorff reflection of the inverse limit of finite T_0 spaces and quotient maps. An explicit construction of $[0, 1]$ as the Hausdorff reflection of an inverse sequence of finite T_0 spaces is also provided; note that in this case the reflector functor does not preserve the limit, since the Hausdorff reflection of a finite space is discrete compact Hausdorff, and thus the inverse limit of the reflections is totally disconnected. It is then natural to ask whether the approach to approximating $[0, 1]$ by means of finite event structures can be generalized. Starting from the consideration that $[0, 1]$ can be seen as a poset-ordered topological space with a basis of order-convex sets, which are taken as the events of the event structure in the construction of Ω, we show than any poset-ordered compact Hausdorff space $(X, \tau, \leq)$ which has a basis of $\leq$ convex sets is homeomorphic to the space of boundaries on the inverse limit of an inverse system of finite event structures, and we argue that this result might provide an alternative way to approximate hyperbolic spacetimes.

References

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