Some Expressive Temporal Logics of Minkowski Spacetimes

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1 On the logical foundations of relativity theories

Classical first-order logic is an expressive logic with strong logical properties. So it is an excellent framework to support axiomatizations and logical analysis of relativity theories. Our project lead by Hajnal Andréka and István Németi has numerous results, some examples are:

- Székely [2013] showed that the existence of superluminal particles is consistent with relativistic kinematics (even in 4D) as long as those particles are not able to coordinatize their environment. This result has been extended to relativistic dynamics as well by Madarász and Székely [2014].
- It is provable from four simple axioms that no observer can move faster than light [Andréka et al. 2006, Cor. 2.1]. This proof was checked by theorem provers as well, which shows another advantage of using formal logic in the foundations of physics. [Govindarajulu, Bringsjord, and Taylor 2014; Stannett and Németi 2014]
- This framework makes it possible to axiomatize special relativity without assuming the structure of real numbers or their first-order theory [Andréka et al. 2002]. Using model-theoretical tools Madarász and Székely [2013] showed that special relativity can be modeled even over the field of rational numbers.
- With a first-order logic analysis it is possible to investigate Why-type questions in physics by studying which axioms are needed and which are superfluous in order to prove certain predictions of relativity theory [Székely 2011]. For instance, Andréka, Madarász, Németi, and Székely [2008] showed that the conservation of mass is not needed to prove the mass-increase theorem.

More on the foundational significance of that project can be found in a recent paper of Friend [2014].

2 Standard systems

The standard formal language of our research group has the advantage that it involves very natural primitives[1]: its basic predicates and relations are

- $+$, $\cdot$, $\leq$, $=$ that refer to the standard mathematical terms,
- $\mathrm{Ph}(b)$: "b is a light signal",
- $\mathrm{Ob}(b)$: "b is an observer/coordinate system",
- $\mathrm{W}(o, b, x, y, z, t)$: "According to the observer/coordinate system o, the body is b located in the spacetime position (x, y, z, t). "

This language fits very well to the spacetime diagram-based language of relativity theories, and makes it possible to build up special and general relativities from very few but still logically (and conceptually) transparent axioms. The standard axiom systems that our group uses and in this report we will frequently will refer to are

- SpecRel, which can derive the basic predictions of special relativity.
- CatRel is a complete extension of SpecRel: all of its models are elementarily equivalent with Minkowski spacetimes where there are no accelerating observers.
- AccRel is an extension of SpecRel where intended models allow accelerating coordinate systems.
- GenRel is reduct of AccRel which is complete w.r.t. Lorentzian manifolds and as such can be considered to be an axiomatization of general relativity.

Andréka, Madarász, Németi, and Székely [2012] gives a nice, short and precise introduction to SpecRel, AccRel and GenRel. The system CatRel is discussed in [Andréka et al. 2007].

3 Operational definitions

The above language of spacetime diagrams, however, is not satisfactory for those who are interested in an *operational foundation of physics,* i.e., in a foundation according to which every basic notion and axiom refers to (simple) *experiments*. We do not know that what does "to be a coordinate

[1] for details see [Andréka et al. 2012].

system" (*Ob*) or "coordinatizing" (*W*) mean *in terms of experiments.* Szabó [2010] and Ax [1978] answered this challenge in special relativity, the former focused on the empirical meanings of coordinate systems and coordinatization process while the latter concentrated on a first-order axiomatic system that reflects operational ideas.

Ax's primitive predicates are

- $a\mathrm{T}p$: "a transmits signal p",
- $a\mathrm{R}p$: "a receives signal p",

These primitives refer to experiments, so any axiom system on that language can be considered to be operational. The only problem was that even if this axiom system is complete w.r.t. Minkowski spacetimes, it was not clear how and exactly how much can be expressed from the paradigmatic effects of special relativity. In other words, the expressive power or definitional properties of Ax's system was unexplored. Andréka and Németi [2014] showed that Ax's system is surprisingly expressive: with the addition of some minor axioms (about selecting a meter rod) makes it *definitionally equivalent* with CatRel. Definitional equivalence means that the two theory are about the same concepts, i.e., they are 'equi-expressive'. The main idea of the proof of this definitional equivalence is that though the language of Ax [1978] seems to be very primitive, numbers, mathematical operations and coordinate systems (the primitives of the language of CatRel) are definable.

But it is still an open question whether similar results can be obtained with accelerating observers or w.r.t general relativity. As far as we see, the ideas of [Andréka and Németi 2014] are not transferable directly to general relativity. Such a result, if there is at all, must be achieved in a radically different way. The main problem is that inertiality of observers does not seem to be definable in that language.

That is where we are now and that is the point where our report steps into the picture. A *corollary* of our main result is a framework that can reproduce the same results (decidability, and completeness w.r.t. Minkowski spacetimes, definitional equivalence with CatRel) such that it can be still considered to be operational. Its primitives are

- $+$, $\cdot$, $\leq$, that refer to the standard mathematical terms
- $\mathrm{P}(e, a, x)$: "in event e, a observes that its clock shows time x".

Contrary to results of Ax and Andréka–Németi this operational attempt does not go into the definition of mathematics in terms of experiments, only the definition of those terms that refer to non-mathematical/physical phenomena. This price was not paid in vane: the definition of inertial (geodetic) observers is possible (and simple) in that language, and by that, the road to acceleration is paved, and the research for an operational axiomatization of *general* relativity is started.

But, as we mentioned, that system is only a corollary of a bigger result. The main result of [Molnár 2015] is that we did this in a way that we linked the remarkable modal research of the literature into our research.

4 Modal perspective

Modal logic, especially *temporal logic* in the foundation of relativity theories are to provide a local perspective of relativistic time and to make the information about spacetime available *in the spacetime itself.*

Modal and temporal logics are, however, usually stay in the propositional level, i.e., no variable bounding quantifiers are used. Instead of these, the common primitive connectives in modal logic are $\Box$ and its dual, $\Diamond$, that stands for 'change': in the semantics, it changes the 'state' or 'model', i.e., the truth of some formulas. According to the relativistic interpretation, 'states' are events and 'change' is the change along the causal evolution of spacetime.

Temporal logics are modal logics where $\Box$ and $\Diamond$ are replaced with a **G** and a **F**, and there is an other pair of modal connectives, **H** and **P**, that makes room for 'memory' in the form of 'backward change'. The relativistic interpretation of these connectives are then

- $\mathbf{G}\varphi$: "φ is always **g**oing to be true in the causal future",
- $\mathbf{F}\varphi$: "φ will be true in the causal **f**uture",
- $\mathbf{H}\varphi$: "φ **h**as always been the case in the causal past",
- $\mathbf{P}\varphi$: "φ was the case in the causal **p**ast",

If the temporal logic in question is propositional, or in other words, a zero-order logic, then this means that its primitive sentences do not bound variables, i.e., has no inner structure; they are just p-s and q-s, that can be true or false but they do not express a relation in the state. They are, however, freely interpretable – that is why we use the expression 'temporal *logic*' instead of 'temporal *theories*'.

To enrich the expressive power of that language, we will use first-order temporal logics instead with the following (familiar) special primitives:

- $+$, $\cdot$ and $\leq$: the standard mathematical terms
- $a{:}\tau$: "clock a shows time τ"

Note that the resemblance with the first-order classical language of the previous section; there is no explicit reference to the events, while the primitive predicate is still operational. Here, a refers to clocks, that x refers to numbers. For a start, a clock is something that – contrary to a number – can change its denotation while the spacetime evolve / the state changes / the time elapse / we shift from one event to another along causality. In that language, a quantification will be local; we can quantify over only those clocks that are actually available:

- $\exists a \varphi$: "there is a clock a here such that φ is true"

We will assume that numbers will always going to be available in every event.

Note that in this language, the clock-relativized temporal operators, and as such the experienced past is immediately definable:

- $\mathbf{P}_a \varphi \overset{\text{def}}{\Leftrightarrow} \mathbf{P}(\exists x\, a{:}x \wedge \varphi)$: "somewhere in the causal past *where a occurred*, φ is true"

Therefore, this system can be considered as a multi-agent system, i.e., a system in which every agent (clock) has its own modal operator. Since we have that local quantification over clocks as well, these agents can talk about each other, they can share information about their past – that is how the exploration of spacetime is look like in this temporal language.

5 Main Results

Some main results of our research so far:

1. **Strong Expressive Power:** Our language can express the basic paradigmatic relativistic effects of kinematics such as time dilation, length contraction, twin paradox, etc.

2. **Strong Axiomatic Base:** The temporal formulas that represent the basic paradigmatic effects of relativity theories can be derived from a finite scheme axiom system SpecClockSys.

3. **Operationality:** The coordinatization itself is definable using *(metric) tense operators* with *signalling procedures.* These operators refer to inertial agents drifting in space and conducting signalling experiments to discover the spacetime they live in. The well-definedness of that coordinatization process is also derivable from SpecClockSys.

4. **Completeness and Decidability:** The true formulas of the acceleration-free 4D Minkowski spacetime can be derived from a finite scheme based axiom system SpecClockSysNoAcc.

5. **Hybrid sort definition:** Nominals, i.e., a hybrid sort $i_0, i_2, \ldots$ can be defined in (connected models of) SpecClockSys, and hybrid operators $@_i$, $\downarrow_i$ and the somewhere operator E is also definable.

6. **Formal comparison to other first-order axiom systems:** Those extensions of SpecRel where there are no FTL bodies and observationally indiscernible bodies are definitionally equivalent with the standard translation of some extension of SpecClockSys. **This means that all classical systems with that property is equivalent to a natural temporal logic**. Specially, the complete CatRel of Andréka, Madarász, and Németi [2007] is definitionally equivalent with the standard translation of SpecClockSysNoAcc.

7. **Incompleteness of unrestricted acceleration in flat spacetimes:** In *flat* Minkowski spacetimes, the existence of certain curves will result in drastic increase of expressive power which results in the interpretability of Robinson-arithmetic **Q** and representability of recursive functions, hence the true formulas of the 4D Minkowski spacetime with *all* (not necessarily inertial) timelike curves are not finite-scheme axiomatizable.

Most details can be found in [Molnár 2015]. In this talk we will overview these results focusing on

- how can we find temporal logical correspondents for classical axiom systems of flat spacetimes, and

- how can we construct *branching spacetimes*, indeterminist spacetimes using these ideas.

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