Epistemic "Holes" in Spacetime*

John Byron Manchak

1 Introduction

A number of models of general relativity seem to contain "holes" which are thought to be "physically unreasonable". One seeks a condition to rule out these models. We examine a number of possibilities already on the table. We then introduce a new condition: epistemic hole-freeness. Epistemic hole-freeness is not just a new condition — it is new in kind. In particular, its motivation is primarily epistemic rather than metaphysical.

2 Preminilaries

We begin with a few preliminaries concerning the relevant background formalism of general relativity.[1] An n-dimensional, relativistic *spacetime* (for $n \geq 2$) is a pair of mathematical objects (M, g_{ab}). M is a connected n-dimensional manifold (without boundary) that is smooth (infinitely differentiable). Here, g_{ab} is a smooth, non-degenerate, pseudo-Riemannian metric of Lorentz signature $(+, -, ..., -)$ defined on M.

Note that M is assumed to be *Hausdorff*; for any distinct $p, q \in M$, one can find disjoint open sets O_p and O_q containing p and q respectively. We say two spacetimes (M, g_{ab}) and (M', g'_{ab}) are *isometric* if there is a diffeomorphism $\varphi : M \to M'$ such that $\varphi_*(g_{ab}) = g'_{ab}$.

For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *timelike* if $g_{ab}\xi^a\xi^b > 0$, *null* if $g_{ab}\xi^a\xi^b = 0$, or *spacelike* if $g_{ab}\xi^a\xi^b < 0$. Null vectors create the "cone structure; timelike vectors are inside the cone while spacelike vectors are

*Thanks to Jeff Barrett, Thomas Barrett, David Malament, Jim Weatherall, and Chris Wüthrich for helpful suggestions on a previous draft.

[1] The reader is encouraged to consult Hawking and Ellis (1973), Wald (1984), and Malament (2012) for details. An outstanding (and less technical) survey of the global structure of spacetime is given by Geroch and Horowitz (1979).

outside. A *time orientable* spacetime is one that has a continuous timelike vector field on M. A time orientable spacetime allows one to distinguish between the future and past lobes of the light cone. In what follows, it is assumed that spacetimes are time orientable.

For some open (connected) interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma : I \to M$ is *timelike* if the tangent vector ξ^a at each point in $\gamma[I]$ is timelike. Similarly, a curve is *null* (respectively, *spacelike*) if its tangent vector at each point is null (respectively, spacelike). A curve is *causal* if its tangent vector at each point is either null or timelike. A causal curve is *future-directed* if its tangent vector at each point falls in or on the future lobe of the light cone.

We say a curve $\gamma : I \to M$ is not *maximal* if there is another curve $\gamma' : I' \to M$ such that I is a proper subset of I' and $\gamma(s) = \gamma'(s)$ for all $s \in I$. A curve $\gamma : I \to M$ in a spacetime (M, g_{ab}) a *geodesic* if $\xi^a \nabla_a \xi^b = \mathbf{0}$ where ξ^a is the tangent vector and ∇_a is the unique derivative operator compatible with g_{ab}.

For any two points $p, q \in M$, we write $p << q$ if there exists a future-directed timelike curve from p to q. We write $p < q$ if there exists a future-directed causal curve from p to q. These relations allow us to define the *timelike and causal pasts and futures* of a point p: $I^-(p) = \{q : q << p\}$, $I^+(p) = \{q : p << q\}$, $J^-(p) = \{q : q < p\}$, and $J^+(p) = \{q : p < q\}$. Naturally, for any set $S \subseteq M$, define $J^+[S]$ to be the set $\cup\{J^+(x) : x \in S\}$ and so on. A set $S \subset M$ is *achronal* if $S \cap I^-[S] = \varnothing$.

A point $p \in M$ is a *future endpoint* of a future-directed causal curve $\gamma : I \to M$ if, for every neighborhood O of p, there exists a point $t_0 \in I$ such that $\gamma(t) \in O$ for all $t > t_0$. A *past endpoint* is defined similarly. A causal curve is *future inextendible* (respectively, *past inextendible*) if it has no future (respectively, past) endpoint.

For any set $S \subseteq M$, we define the *past domain of dependence of* S, written $D^-(S)$, to be the set of points $p \in M$ such that every causal curve with past endpoint p and no future endpoint intersects S. The *future domain of dependence of* S, written $D^+(S)$, is defined analogously. The entire *domain of dependence of* S, written $D(S)$, is just the set $D^-(S) \cup D^+(S)$. The *edge* of an achronal set $S \subset M$ is the collection of points $p \in S$ such that every open neighborhood O of p contains a point $q \in I^+(p)$, a point $r \in I^-(p)$, and a timelike curve from r to q which does not intersect S. A set $S \subset M$ is a *slice* if it is closed, achronal, and without edge. A spacetime (M, g_{ab}) which contains a slice S such that $D(S) = M$ is said to be *globally hyperbolic.*

3 A Condition to Disallow Holes?

Consider the following example.

Example 1. Let (M, g_{ab}) be Minkowski spacetime and let p be any point in M. Consider the spacetime $(M - \{p\}, g_{ab})$.

The spacetime seems to have an artificial "hole" (see Figure 1). One seeks to find a (simple, physically meaningful) condition to disallow the example. (The condition need not be a sufficient condition for "physical reasonableness"; it need only be necessary.) But "although one perhaps has a good intuitive idea of what it is that one wants to avoid, it seems to be difficult to formulate a precise condition to rule out such examples" (Geroch and Horowitz 1979, 275).

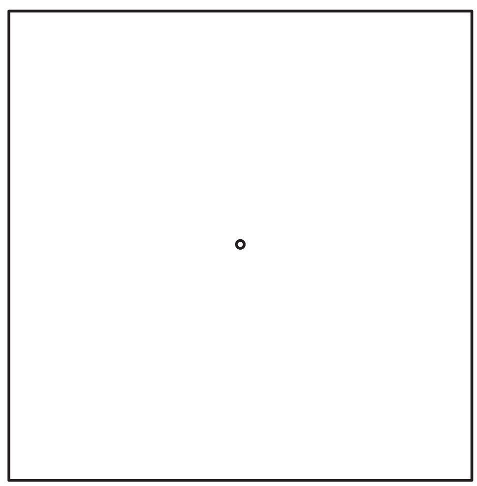

Figure 1: Minkowski spacetime with one point removed.

Many of the conditions used to rule out the "hole" in example 1 require that certain regions of (or curves in) spacetime be "as large as they can be". For example, geodesic completeness requires every geodesic to be as large as it can be in a certain sense. Hole-freeness essentially requires the domain of dependence of every spacelike surface to be as large as it can be. Inextendibility requires the entirety of spacetime to be as large as it can be. Let us examine each of these three conditions in more detail. First, consider geodesic completeness.

Definition. A spacetime (M, g_{ab}) is *geodesically complete* if every maximal geodesic $\gamma : I \to M$ is such that $I = \mathbb{R}$. A spacetime is *geodesically incomplete* if it is not geodesically complete.

If an incomplete geodesic is timelike or null, there is a useful distinction one can introduce (which we will need later). We say that a future-directed timelike or null geodesic $\gamma : I \to M$ without future endpoint is *future incomplete* if there is an $r \in \mathbb{R}$ such that $s < r$ for all $s \in I$. A *past incomplete* timelike or null geodesic is defined analogously. Next, consider inextendibility.

Definition A spacetime (M, g_{ab}) is *extendible* if there exists a spacetime (M', g'_{ab}) and an isometric embedding $\varphi : M \to M'$ such that $\varphi(M) \subsetneq M'$. Here, the spacetime (M', g'_{ab}) is an *extension* of (M, g_{ab}). A spacetime is *inextendible* if has no extension.

Finally, consider hole-freeness. Initially, one defined (Geroch 1977) a spacetime (M, g_{ab}) to be *hole-free* if, for every spacelike surface $S \subset M$ and every isometric embedding $\varphi : D(S) \to M'$ into some other spacetime (M', g'_{ab}), we have $\varphi(D(S)) = D(\varphi(S))$. The definition seemed to be satisfactory. But surprisingly, it turns out the definition is too strong; Minkowski spacetime fails to be hole-free under this formulation (Krasnikov 2009). But one can make modifications to avoid this consequence (Manchak 2009).

Let (K, g_{ab}) be a globally hyperbolic spacetime. Let $\varphi : K \to K'$ be an isometric embedding into a spacetime (K', g'_{ab}). We say (K', g'_{ab}) is an *effective extension* of (K, g_{ab}) if, for some Cauchy surface S in (K, g_{ab}), $\varphi[K] \subsetneq int(D(\varphi[S]))$ and $\varphi[S]$ is achronal. Hole-freeness can then be defined as follows.

Definition. A spacetime (M, g_{ab}) is *hole-free* if, for every set $K \subseteq M$ such that $(K, g_{ab|K})$ is a globally hyperbolic spacetime with Cauchy surface S, if $(K', g_{ab|K'})$ is not an effective extension of $(K, g_{ab|K})$ where $K' = int(D(S))$, then there is no effective extension of $(K, g_{ab|K})$.

What is the relationship between the three conditions? There are only two implication relations between them (see Manchak 2014 for all proofs and counterexamples).

Proposition. Any spacetime which is geodesically complete is (i) hole-free and (ii) inextendible.

Now, any of the three conditions can be used to rule out the "hole" in example 1. But due to the singularity theorems (Hawking and Penrose 1970), geodesic completeness is now considered to be much too strong a condition; it seems to be violated by physically reasonable spacetimes. In what follows, let us focus on the remaining two conditions which are usually taken to be satisfied by all physically reasonable spacetimes. Indeed, these two conditions are still in use (see Earman 1995). Might hole-freeness or inextendibility (or their conjunction) be the condition we are looking for? Consider the following example.

Example 2. Let (M, g_{ab}) be Minkowski spacetime and let p be any point in M. Let $\Omega : M - \{p\} \to \mathbb{R}$ be a smooth positive function which approaches infinity as the point p is approached. Now consider the spacetime $(M - \{p\}, \Omega^2 g_{ab})$.

The spacetime in example 2 is geodesically complete. It is therefore inextendible and hole-free. Nonetheless, it seems there is still an artificial "hole" in the spacetime. One seeks a (simple, physically meaningful) condition to rule out even these holes.

4 A New Condition

Consider the following definition.

Definition. A spacetime (M, g_{ab}) has an *epistemic hole* if there is a point $p \in M$ and two future-inextedible timelike curves γ and γ' through p such that $I^-[\gamma]$ is a proper subset of $I^-[\gamma']$.

The physical significance of the definition is this: Suppose you and I are both present at some event. Now suppose you go your way and I go mine. If it is the case that I can eventually know everything you can eventually know *and more*, then there is a kind of epistemic "hole" preventing you from knowing the extra bit. One might require the region of spacetime which an observer can eventually know to be "as large as it can be". In other words, one might require spacetime to be free from epistemic holes.

Is the definition adequate? Examples 1 and 2 have epistemic holes as we would expect. But, unfortunately, the condition is too strong; it rules

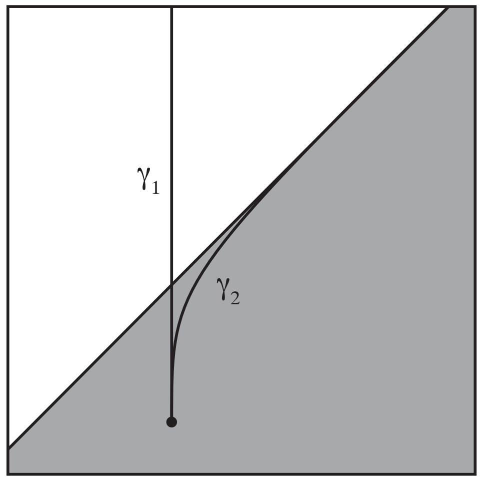

Figure 2: Observers γ_1 and γ_2 in Minkowski spacetime. The set $I^-[\gamma_2]$ (the shaded area) is a proper subset of $I^-[\gamma_1]$ (the entire manifold).

out spacetimes which are usually thought to be physically reasonable is some sense. Take Minkowski spacetime, for example. It has epistemic holes. (Consider any point in the Minkowski spacetime. Now consider any observer at the point who accelerates to reach "null infinity" and another observer at the point who does not. See Figure 2.)

But there is no serious problem here; we can restrict attention to timelike geodesics rather than arbitrary timelike curves. (We will use the definition below rather than the definition above in what follows.)

Definition. A spacetime (M, g_{ab}) has an *epistemic hole* if there is a point $p \in M$ and two future-inextedible timelike geodesics γ and γ' through p such that $I^-[\gamma]$ is a proper subset of $I^-[\gamma']$.

Despite the relaxed formulation, examples 1 and 2 still count as having epistemic holes (see Figure 3). Moreover, a number of spacetimes thought to be physically reasonable are epistemically hole-free (e.g. Minkowski spacetime). The condition of epistemic hole-freeness is somewhat permissive in that it does not automatically rule out acausal spacetimes (e.g. Gödel spacetime).

On the other hand, some spacetimes with "naked singularities" have epistemic holes (e.g. Misner spacetime). But spacetimes with naked singularities

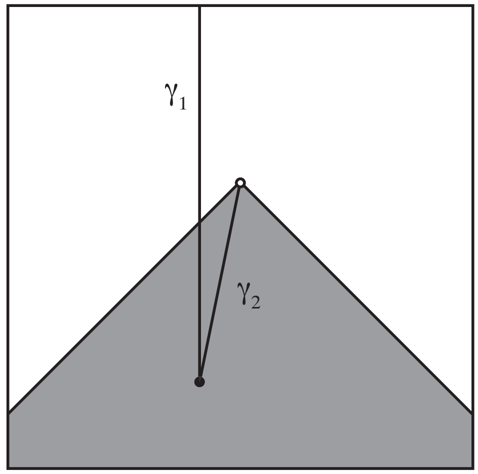

Figure 3: Observers γ_1 and γ_2 in Minkowski spacetime with one point removed. The set $I^-[\gamma_2]$ (the shaded area) is a proper subset of $I^-[\gamma_1]$ (the entire manifold).

are thought to be physically unreasonable in some sense. Consider the following influential definition (Geroch and Horowitz 1979, Earman 1995).

Definition. A spacetime (M, g_{ab}) has a *naked singularity* if there is a point $p \in M$ and a future-incomplete timelike geodesic γ such that $\gamma \subseteq I^-(p)$.

What is the relationship between naked singularities and epistemic holes? Example 2 shows that naked singularities are not equivalent to epistemic holes (the example contains no naked singularities). And the following example contains naked singularities but no epistemic holes.

Example 3. Let (M, g_{ab}) be two dimensional Minkowski spacetime in standard t, x coordinates which is "rolled up" along the t direction. Let p be any point in M. Consider the spacetime $(M - \{p\}, g_{ab})$.

Now, the condition of global hyperbolicity is sufficient to exclude naked singularities (Geroch and Horowitz 1979). In addition, global hyperbolicity together with inextendibility requires spacetime to be hole-free (Manchak 2009). And, despite the fact that the global hyperbolicity is a strong causal condition, some hold that it is satisfied by all physically reasonable spacetimes (Penrose 1979). Might it be the case that global hyperbolicity rules out epistemic holes? Consider the following.

Example 4. Let (M, g_{ab}) be Minkowski spacetime and let p be any point in M. Let M' be the set $I^-(p)$. Let $\Omega : M' \to \mathbb{R}$ be a smooth positive function which approaches infinity as the boundary of $I^-(p)$ is approached. Now consider the spacetime $(M', \Omega^2 g_{ab})$. (See Figure 4.)

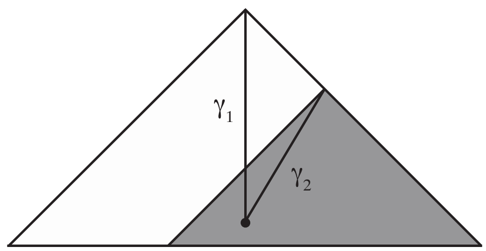

Figure 4: Observers γ_1 and γ_2 in a portion of conformal Minkowski spacetime. The set $I^-[\gamma_2]$ (the shaded area) is a proper subset of $I^-[\gamma_1]$ (the entire manifold).

Example 4 shows that a globally hyperbolic spacetime, indeed *even a globally hyperbolic spacetime which is geodesically complete*, can nonetheless have epistemic holes. On the other hand, example 3 shows that a spacetime which is non-globally hyperbolic, indeed *even a non-globally hyperbolic spacetime which fails to be inextendible and hole-free*, can nonetheless be epistemically hole-free. In sum: epistemic holes are very different from "holes" and "singularities" of various kinds.

5 A New Type of Condition

Stepping back, we note that the definition of epstemic hole-freeness differs from (and indeed is more attractive than) inextendibility and hole-freeness in two important ways.

First, inextendibility and hole-freeness require that certain regions of spacetime be "as large as they can be" in the sense that one compares them, from a God's eye point of view, to similar regions in *all possible spacetimes*. And without knowing in advance which of all possible spacetimes are physically reasonable (and how could one know?) such a comparison is on unsteady ground. But consider epistemic hole-freeness. The condition requires that certain regions of spacetime be "as large as they can be" in the sense that one compares them to similar regions *within the very same spacetime.* Thus, we have one way in which the proposed condition is a bit easier to swallow (and pragmatically easier to work with).

Second, there is a sense in which, if inextendibility or hole-freeness are satisfied or violated in a spacetime, observers in the spacetime may not have the epistemic resources to know it. But this is not the case for epistemic holes. Consider the following definition (Glymour 1977).

Definition. Two space-times (M, g_{ab}) and (M', g'_{ab}) are *observationally indistinguishable* if for every future-inextendible timelike curve γ in (M, g_{ab}), there is a future-inextendible timelike curve γ' in (M', g'_{ab}) such that $I^-[\gamma]$ and $I^-[\gamma']$ are isometric; and, correspondingly, with the roles of (M, g_{ab}) and (M', g'_{ab}) interchanged.

Let us say that a spacetime property is *preserved under observational indistinguishability* if, for every pair of observationally indistinguishable spacetimes, one has the property just in case the other does as well. We have the following.

Proposition. Epistemic hole-freeness is preserved under observational indistinguishability. Inextendibility and hole-freeness are not.

Proof. Let (M, g_{ab}) and (M', g'_{ab}) be observationally indistinguishable spacetimes. Suppose (M, g_{ab}) has an epistemic hole. So, there is a point $p \in M$ and two future-inextedible timelike geodesics γ and γ' through p such that $I^-[\gamma]$ is a proper subset of $I^-[\gamma']$. Since (M, g_{ab}) and (M', g'_{ab}) are observationally indistinguishable, there is a future-directed timelike curve λ in (M', g'_{ab}) and an isometry $\varphi : I^-[\gamma'] \to I^-[\lambda]$. Clearly, it must be the case that $\varphi[\gamma]$ and $\varphi[\gamma']$ are future-inextedible timelike geodesics through $\varphi(p)$ such that $I^-[\varphi[\gamma]]$ is a proper subset of $I^-[\varphi[\gamma']]$. In other words, (M', g'_{ab}) has an epistemic hole. The other direction is analogous. So epistemic hole-freeness is preserved under observational indistinguishability.

Next, consider hole-freeness. Let (M, g_{ab}) be two dimensional Minkowski spacetime in standard t, x coordinates with the set $\{(t, x) : t \geq 0\}$ removed from the manifold. Let p be any point in M and let M' be the manifold $I^-(p)$. One can verify that (M, g_{ab}) and (M', g_{ab}) are observationally indistinguishable. But (M, g_{ab}) is not hole-free and (M', g_{ab}) is. So hole-freeness is not preserved under observational indistinguishability (see Figure 5).

For the inextendibility case, see Malament (1977, 78). □

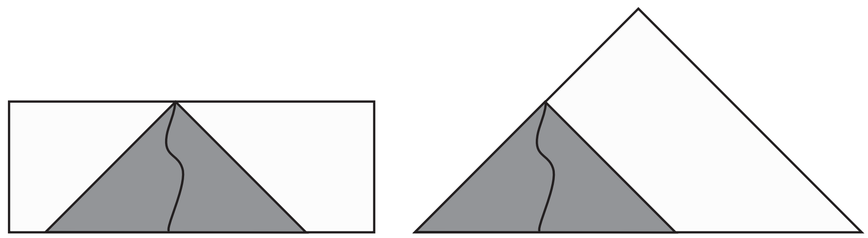

Figure 5: Two portions of Minkowski spacetime. Only the one to the right is hole-free. The spacetimes are observationally indistinguishable since all pasts of all observers in both spacetimes (shaded regions) are isometric.

The proposition shows another sense in which epistemic hole-freeness is more appropriate to presuppose than hole-freeness or inextendibility. If it seems to an observer that her spacetime is hole-free and inextendible, this gives little reason to be confident that the spacetime actually satisfies the conditions. In fact, confidence in the satisfaction of these two conditions comes primarily from metaphysical principles involving plenitude and causal determinism (Earman 1995, Manchak 2011). This should give us pause. On the other hand, if it seems to an observer that her spacetime is epistemically hole-free, this *does* give some reason to be confident that the spacetime actually satisfies the condition.

References

[1] Earman, J. (1995). *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetimes.* Oxford: Oxford University Press.

[2] Geroch, R. (1977), "Prediction in General Relativity," *Foundations of Space-Time Theories, Minnesota Studies in the Philosophy of Science* Vol. VIII, J. Earman, C. Glymour, and J. Statchel (eds.), 81-93. Minneapolis: University of Minnesota Press.

[3] Geroch, R., and Horowitz, G. (1979). "Global Structure of Spacetimes," in S. W. Hawking and W. Israel (eds), *General Relativity: An Einstein Centenary Survey.* Cambridge: Cambridge University Press, 212-293.

[4] Glymour, C. (1977). "Indistinguishable Spacetimes and the Fundamental Group," in J. Earman, C. Glymour, and J. Stachel (eds), *Foundations of Space-Time Theories, Minnesota Studies in the Philosophy of Science Vol. VIII.* Minneapolis: University of Minnesota Press, 50-60.

[5] Hawking, S., and Penrose, R. (1970). "The Singularities of Gravitational Collapse and Cosmology," *Proceedings of the Royal Society of London A*, 314, 529-548.

[6] Hawking, S. and G. Ellis (1973), *The Large Scale Structure of Space-Time.* Cambridge: Cambridge University Press.

[7] Krasnikov, S. (2009). "Even the Minkowski Space is Holed," *Physical Review D* 79: 124041.

[8] Malament, D. (1977). "Observationally Indistinguishable Spacetimes," in J. Earman, C. Glymour, and J. Stachel (eds), *Foundations of Space-Time Theories, Minnesota Studies in the Philosophy of Science Vol. VIII.* Minneapolis: University of Minnesota Press, 61-80.

[9] Malament, D. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory.* Chicago: University of Chicago Press.

[10] Manchak, J. (2009). "Is Spacetime Hole-Free?" *General Relativity and Gravitation*, 41: 1639-1643.

[11] Manchak, J. (2011). "What is a Physically Reasonable Spacetime?" *Philosophy of Science*, 78: 410-420.

[12] Manchak, J. (2014). "On Spacetime Singularities, Holes, and Extensions" *Philosophy of Science*, 81: 1066-1076.

[13] Penrose, R. (1979). "Singularities and Time-Asymmery," in S. Hawking and W. Israel (eds), *General Relativity: An Einstein Centenary Survey.* Cambridge: Cambridge University Press, 581-638.

[14] Wald, R. (1984). *General Relativity.* Chicago: University of Chicago Press.