Guarded Fragment Of First Order Logic Without Equality

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Abstract

Let n be finite. Let L_n denotes the n-variable first order logic without equality, whose semantics are provided by generalized models, where assignments are allowed from arbitrary sets of n-ary sequences $V \subseteq {}^nU$ (some non empty set U). Semantics for the Boolean connectives are defined the usual way, and for the existential quantifier $\exists x_i$ $(i < n)$, a generalized model V and $s \in V$, $V, s \models \exists x_i \phi \iff (\exists s)(s \equiv_i t) \& V, s \models \phi$. We show that L_n is complete, sound, decidable, has the finite model property, the Craig interpolation and the Beth definability properties, but has the Gödel's incompleteness property. Our proof is algebraic addressing the classes Crs_n^{df}'s of diagonal free reducts of Crs_n's. The positive results are proved by showing that Crs_n is a variety that is finitely axiomatizable, that the universal theory of Crs_n^{df} is decidable, that Crs_n^{df} has the finite base property, and finally that it has the super amalgamation property. Gödels's incompleteness is proved by showing that the free algebras with at least one (free) generator are not atomic.

1 Results

In this shortened version of the paper, we present only the positive results. The result on Gödel's incompleteness is omitted. Throughout the paper, unless othewise indicated, n is a finite ordinal. For a set X, $\mathfrak{B}(X)$ is the Boolean set algebra with top element X, that is $\mathfrak{B}(X) = \langle \wp(X), \cup, \cap, \sim \rangle$. Let U be a non–emptyset and $V \subseteq {}^nU$. For $X \subseteq V$ and $i < n$, we set $\mathsf{C}_i X = \{t \in {}^\alpha U : \exists s \in X s \equiv_i t\}$. We denote the class of set algebras of the form $\langle \mathfrak{B}(V), \mathsf{C}_i \rangle_{i,j<n}$ where V is an arbitary set of n–ary sequences, by Crs_n^{df}.

Theorem 1.1. *Crs_n^{df} is finitely axiomatizable variety, and the universal theory of Crs_n^{df} is decidable.*

Proof. Crs_n^{df} is axiomatized by the axioms of the cylindric algebras, CA_n, except the ones dealing with diagonals and the axiom stating that the cylindrifiers commute with each other. Our proof uses the games and networks as introduced to algebraic logic by Hirsch and Hodkinson. However, the step by step method, due to Andreka, can be used to obtain the same axiomatization.

For decidability, one proof can be distilled from the more general [2, Theorem 4]. Another proof: For $\mathfrak{A} \in Crs_n^{df}$, let $(\mathfrak{A})$ be the first order signature consisting of an n–ary relation symbol for each element of $\mathfrak{A}$. Then we show that for every $\mathfrak{A} \in Crs_n^{df}$, for any $\psi(x)$ a quantifier free formula of the signature of Crs_n^{df} and $\bar{a} \in \mathfrak{A}$ with $|\bar{a}| = |\bar{x}|$, there is a loosely guarded $(\mathfrak{A})$ sentence $\tau_{\mathfrak{A}}(\psi(\bar{a}))$ whose relation symbols are among $\bar{a}$ such that for any relativized representation M of $\mathfrak{A}$, $\mathfrak{A} \models \psi(\bar{a}) \iff M \models \tau_{\mathfrak{A}}(\psi(\bar{a}))$. Let $\mathfrak{A} \in Crs_n^{df}$ and $\bar{a} \in \mathfrak{A}$. We start by the terms. Then by induction we complete the translation to quantifier free formulas. For any tuple $\bar{u}$ of distinct n variables, and term $t(\bar{x})$ in the signature of Crs_n^{df}, we translate $t(\bar{a})$ into a loosely guarded formula $\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a}))$ of the first order language having signature $L(\mathfrak{A})$. If t is a variable, then $t(\bar{a})$ is a for some $a \in \mathsf{rng}(\bar{a})$, and we let $\tau_{\mathfrak{A}}^{\bar{u}}(t(a)) = a(\bar{u})$.

Now assume inductively that $t(\bar{a})$ and $t'(\bar{a})$ are already translated. We suppress $\bar{a}$ as it plays no role here. For all $i < n$, define (for the clause c_i, w is a new variable):

$$\begin{aligned}
\tau_{\mathfrak{A}}^{\bar{u}}(-t) &= 1(\bar{u}) \wedge \neg\tau_{\mathfrak{A}}^{\bar{u}}(t),\\
\tau_{\mathfrak{A}}^{\bar{u}}(t+t') &= \tau_{\mathfrak{A}}^{\bar{u}}(t) + \tau_{\mathfrak{A}}^{\bar{u}}(t),\\
\tau_{\mathfrak{A}}^{\bar{u}}(\mathsf{c}_i t) &= 1(\bar{u}) \wedge \exists w[(1(\bar{u}_w^i) \wedge \tau_{\mathfrak{A}}^{\bar{u}_w^i}(t)],
\end{aligned}$$

Let M be a relativized representation of $\mathfrak{A}$, then $\mathfrak{A} \models t(\bar{a}) = t'(\bar{a}) \iff M \models \forall\bar{u}[\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a})) \longleftrightarrow \tau_{\mathfrak{A}}^{\bar{u}}(t'(\bar{a}))]$. For terms $t(\bar{x})$ and $t'(\bar{x})$ and $\bar{a} \in \mathfrak{A}$, choose pairwise distinct variables $\bar{u}$, that is for $i < j < n$, $u_i \neq u_j$ and define $\tau_{\mathfrak{A}}(t(\bar{a}) = t'(\bar{a})) := \forall\bar{u}[1(\bar{u}) \to (\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a})) \longleftrightarrow \tau_{\mathfrak{A}}^{\bar{u}}(t'(\bar{a})))]$. Now extend the definition to the Boolean operations as expected, thereby completing the translation of any quantifier free formula $\psi(\bar{a})$ in the signature of Crs_n^{df} to the $(\mathfrak{A})$ formula $\tau_{\mathfrak{A}}(\psi(\bar{a}))$.

Then it is easy to check that, for any quantifier free formula $\psi(\bar{x})$ in the signature of Crs_n^{df} and $a \in \mathfrak{A}$, we have:

$$\mathfrak{A} \models \psi(\bar{a}) \iff M \models \tau_{\mathfrak{A}}(\psi(\bar{a})),$$

and the last is a loosely guarded $(\mathfrak{A})$ sentence.

By decidability of the loosely guarded fragment the required result follows. □

Theorem 1.2. *Crs_n^{df} has the finite model property*

Proof. **First proof:** Assume $\mathfrak{A} \models \neg\psi(\bar{a})$ for some $\bar{a} \in {}^n\mathfrak{A}$. We want to find $\mathfrak{B}$ that falsfies ψ. We may assume, without loss of generality, that $0, 1 \in \mathsf{rng}(\bar{a})$.

Let M be a relativized representation of $\mathfrak{A}$, then $M \models \tau_{\mathfrak{A}}(-\psi(\bar{a}))$ and all relation symbols in $\tau_{\mathfrak{A}}(-\psi(\bar{a}))$ are from $\bar{a}$. Let $\rho = \forall \bar{u}(a(\bar{u}) \to 1^{\mathfrak{A}}(\bar{u}))$ for every relation symbol a in $\bar{a}$. Then ρ is logically equivalent to a loosely guarded $L(\mathfrak{A})$ sentence and $M \models \rho$. Now $\rho \wedge \tau_{\mathfrak{A}}$ is equivalent to a loosely guarded $L(\mathfrak{A})$ sentence too, and $M \models \rho \wedge \tau_{\mathfrak{A}}(\psi(a))$. Hence there is a finite model $N \models \rho \wedge \tau_{\mathfrak{A}}(\neg\psi(\bar{a}))$. Define $\mathfrak{B}$ to be the set algebra where everything is relativized to N, that is $\mathfrak{B}$ has universe $\wp((1^{\mathfrak{A}})^N)$.

Second proof: [2, Theorem 4].

Here we use the following model–theoretic result of Herwig's [2].

(*) Let R be a finite structure in a finite relational language L. Then, there is a finite L structure $\mathsf{R}^+ \supseteq \mathsf{R}$ such that any partial isomorphism of R is induced by an automorphism of R^+

Using (*), we give the main steps of the proof tailored to the special case Crs_n^{df}. Let $\mathfrak{F}$ be any finite set of terms in the language of Crs_n. Let $\mathfrak{A}$ be a given set algebra, that is $\mathfrak{A} \in Crs_n$ and let u be an assignment of the variables in $\mathfrak{F}$ to elements of $\mathfrak{A}$. It suffices to construct an algebra $\mathfrak{B}$ with finite base and an assignment v of the same variables to elements of $\mathfrak{B}$, such that for all τ and $\sigma \in \mathfrak{F}$ we have, cf. [2, Theorem 4, p.248]:

$$\mathfrak{A} \models \tau = \sigma[u] \iff \mathfrak{B} \models \tau = \sigma[v].$$

We refer to the latter statement by $(*)$.

Let U be the base of $\mathfrak{A}$ Let $\mathfrak{U}$ be the first order structure with domain U in a signature t determined as follows. For every $\tau \in \mathfrak{F}$, there is a relation symbol $\bar{\tau} \in t$. If τ does not begin with c_i then the arity of $\bar{\tau}$ is n. If τ begins with c_i then the arity of $\bar{\tau}$ is $n \sim \{i\}$. The relation symbol τ is interpreted as $\tau^{\mathfrak{A},u}$ if the arity of τ is n. Otherwise it is interpreted as $\{s \in n \sim \{i\} : s \in \tau^{\mathfrak{A},u}\}$.

Let Q be an arbitrary finite subset of U. For any $s \in {}^nU$, define a finite substructure $\mathfrak{U}(s)$ of $\mathfrak{U}$ domain $Q \cup \mathsf{rng}(s)$, expanded with n constants $c_i : i < n$ with s_i interpreting c_i. The structures $\mathfrak{U}(s)$ have at most $|Q| + n$ elements and their signature is finite, hence their isomorphism types are finite. Hence it is possible to find a finite $\mathsf{K} \subseteq U$ such that for all $s \in {}^nU$, there is a $t \in {}^n\mathsf{K}$, with $\mathfrak{U}(t) \cong \mathfrak{U}(s)$.

Fix such K. Let R be the substructure of U with domain K. It has a finite relational signature, so we obtain by Herwig's theorem a finite $\mathfrak{F}^*$ structure R^+ such that any partial isomorphism of R is induced by an automorphism of R^+. Let G be the group of automorphisms of R^+ that fix Q point-wise, that is: $G = \{g \in \mathsf{Aut}\mathsf{R}^+ : xg = x \text{ for all } x \in Q\}$. One then define the unit of the required algebra via $H = \{sg : s \in 1^{\mathsf{R}}, g \in G\}$. We now obtain an algebra $\mathfrak{B}$

with universe $\wp(H)$. Let $v(y) = \{s \in H : \mathsf{R}^+ \models \bar{y}(s)\}$. hen v satisfies $(*)$, cf. [2], p.249-252.

$\square$

Now let α be an arbitrary ordinal. We show that Crs_α^{df} has the super amalgamation property ($SUPAP$). For a set X, let $B(X)$ denote the Boolean algebra $(\wp(X), \cap, \sim)$.

Definition 1.3. A *frame* of type Crs_α^{df} is a first order structure $\mathfrak{F} = (X, T_i)_{i\in\alpha}$, where X is an arbitrary set and and $T_i \subseteq X \times X$ for all $i \in \alpha$. Given a frame $\mathfrak{F} = (X, T_i)_{i<\alpha}$, its complex algebra denote by $\mathfrak{Cm}\mathfrak{F}$ is the algebra $(B(X), \mathsf{c}_i)_{i<\alpha}$ where for $Y \subseteq X$, $\mathsf{c}_i(Y) = \{\mathsf{s} \in X : \exists t \in Y, (t, s) \in T_i\}$. Given K having the same signatue as Crs_α, $Str(K) = \{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in K\}$.

(2) Given a family $(\mathfrak{F}_i)_{i\in I}$ a *zigzag product* of these frames is a substructure S of $\prod_{i\in I} \mathfrak{F}_i$ such that the projection maps restricted to S are surjective

Theorem 1.4. *The variety Crs_α^{df} has $SUPAP$.*

Proof. The tecniques in [6] proving the strong amalgamation for Crs_α work verbatim for Crs_α^{df}. But we want $SUPAP$ which is (strictly) stronger. We proceed as follows. Since Crs_α^{sf} is defined by positive equations then it is canonical. In this case, $\mathbf{K} = Str(Crs_\alpha^{df}) = At(Crs_\alpha^{df})$ consists of frames of the form $(V, T_i)_{i<\alpha}$ where V is an arbitrary set of α–ary sequences and for $s, t \in V$, $(s, t) \in T_i \iff s(j) = t(j)$ for all $j \neq i$. Clearly $\mathbf{K}$ closed under forming subalgebras and finite products, hence closed under zigzag products, so by [5, Lemma 5.2.6 p.107], Crs_α^{df} has $SUPAP$. $\square$

Question: Is the equational theory of Crs_ω^{df} decidable?

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