Classification of absorbent-continuous, densely ordered and complete, group-like FL_e-chains

Sándor Jenei
Institute of Mathematics and Informatics
University of Pécs
Ifjúság u. 6, H-7624 Pécs, Hungary
jenei@ttk.pte.hu

The main result of the talk is the classification of absorbent-continuous, order-dense and complete, group-like FL_e-chains: Every such algebra can be described as the twin rotation of a certain BL-algebra and its de Morgan dual (Theorem 5 below). This theorem largely generalizes the corresponding main theorem of [21, 22] and the complexity and length of its proof is significantly reduced, too. Further generalization of this result is not possible; if any of the four assumptions, namely order density, completeness, the group-like property, or absorbent-continuity is dropped then there exist algebras with different form than that of Theorem 5.

The main goal of the talk is to demonstrate how underlying *geometric* ideas and arguments result in proving such a theorem.

Residuation is a basic concept in mathematics [4]. It is strongly connected with Galois maps [14] and closure operators. Residuated *semigroups* have been introduced in the 30s of the last century by Ward and Dilworth [28] to investigate ideal theory of commutative rings with unit. Examples of residuated lattices include Boolean algebras, Heyting algebras [26], complemented semigroups [7], bricks [5], residuation groupoids [8], semiclans [6], Bezout monoids [3], MV-algebras [9], BL-algebras [15], and lattice-ordered groups; a number of other algebraic structures can be rendered as residuated lattices. As witnessed by a more recent and flourishing research field, residuated lattices play a crucial role also in the algebraization of substructural logics. A few examples of substructural logics are classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic, along with their non-commutative versions. These logics had different motivations and methodology, and it was the theory of residuated lattices, which made possible for the theory of substructural logics to cover all these logics (and many others) by the same motivational and methodological umbrella [13]. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science.

As for the *classification* of residuated lattices, as one naturally expects, this is possible only by imposing additional postulates. A first precursor is due to Hölder, who proved in [16] that every *cancellative*, *Archimedean*, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers. Aczél used tools of analysis to investigate continuous semigroup operations over intervals of real numbers[1] and also found in [1, page 256] the cancellative property[2] to be sufficient and necessary for the existence of an order-isomorphism to a subsemigroup of the additive semigroup of the real numbers [1, page 268]. Clifford showed in [10] that every *Archimedean*, naturally and totally ordered semigroup in which the *cancellation law does not hold* can be embedded into either the real numbers in the interval $[0, 1]$ with the usual ordering and $ab = \max(a + b, 1)$ or the real numbers in the interval $[0, 1]$ and the symbol ∞ with the usual ordering and $ab = a + b$ if $a + b \leq 1$ and $ab = \infty$ if $a + b > 1$. For a summary of the Hölder and Clifford theorems, see [11, Theorem 2 in Section 2 of Chapter XI]. Clifford introduced also the ordinal sum construction for a family of totally ordered semigroups in [10] and proved that every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible semigroups of this kind. Mostert and Shields gave a complete description of topological semigroups over compact manifolds with connected, regular boundary in [27] by using a subclass of compact connected Lie groups and via classifying semigroups on arcs such that one endpoint is an identity for the semigroup, and the other is a zero. They classified such semigroups as ordinal sums of three basic multiplications which an arc may possess. The word 'topological' refers to the continuity of the semigroup operation with respect to the topology. In the next related classification result, the property of topological connectedness of the underlying chain was dropped whereas the continuity condition was somewhat strengthened: Under the assumption of divisibility[3][4], which is the dual notion of being naturally ordered, residuated chains were classified as ordinal sumsof linearly ordered Wajsberg hoops in [2]. Postulating the divisibility condition proved to be sufficient for the classification of commutative, integral, prelinear residuated monoids over arbitrary lattices, see [25], where the authors introduced the notion of poset sum of hoops, a common generalization of ordinal sum and of direct product. They proved that certain GBL-algebras embed into the poset sum of a family of MV-chains and that the embedding is an isomorphism in the finite case. Note that BL-algebras are particular commutative GBL-algebras. Next, SIU-chains were classified in [23]. Here the authors assume the existence of a dual isomorphism between the positive and negative cones of the algebra. For SIU-algebras over complete, order dense chains, this condition is equivalent to postulating divisibility only for the negative cone of the algebra. Next, absorbent-continuous, order-dense, group-like FL_e-algebras over complete, weakly real chains were classified in [21, 22]. Although absorbent-continuity looks much weaker than continuity (divisibility) in the negative cone (it only asserts continuity at a subset of the negative cone whose closure has measure zero if, e.g., $[0, 1]$ is considered), in the case

[1] Isotonicity of the semigroup operation is not assumed.

[2] He called it reducible.

[3] For residuated integral monoids, divisibility is equivalent to the continuity of the semigroup operation in the order topology if the underlying chain is order dense.

[4] Divisibility is the algebraic analogue of the Intermediate Value Theorem in real analysis, and for residuated integral monoids over order dense chains, it can be considered as a stronger version of continuity of the monoid operation than the continuity of it with respect to the order topology. Indeed, divisibility entails continuity on order dense chains as mentioned in the previous footnote. On the other hand, if the order topology of the chain is discrete then every operation is continuous but obviously not all operations obey the divisibility condition.

of group-like FL_e-algebras over complete, weakly real chains the two conditions are equivalent. All these lead to our main theorem, Theorem 5. None of the conditions of Theorem 5 can be dropped such that we obtain the same class of algebras.

We shall recall the definitions, which are needed to state our theorem.

- A groupoid $(X, \circledast)$ is a set X with a binary operation $\circledast$.
- A partially ordered monoid (po-monoid in the sequel) is *integral* (resp. *dually integral*) if it has a top (resp. bottom) element which is also the unit element of the operation.
- A commutative binary operation $\circledast$ on a poset $(X, \leq)$ is called *residuated* if there exists another binary operation $\rightarrow_\circledast$ on X such that for $x, y, z \in X$, $x \circledast y \leq z$ iff $y \rightarrow_\circledast z \geq x$.
- Call $\mathcal{U} = (X, \circledast, \leq, t, f)$ an *FL_e-monoid* if $\mathcal{C} = (X, \leq)$ is a poset, $(X, \circledast)$ is a commutative, residuated monoid over $\mathcal{C}$ with neutral element t, and f is an arbitrary constant. Call $\mathcal{U}$ an *FL_e-algebra* if $\leq$ is a lattice order. *Commutative residuated lattices* are exactly the f-free reducts of FL_e-algebras ([13]).
- Call $\mathcal{U}$ *involutive*, if for $x \in X$, $(x')' = x$ holds, where $x' = x \rightarrow_\circledast f$. For involutive FL_e-monoids (referring to $f' = t$) one extremal case is the integral case, that is, when t is the top element and f is the bottom element. The other extremal case is the group-like case, when t and f coincide: We call an FL_e-monoid *group-like*, if it is involutive and $t = f$ holds[5].
- Define the positive and the negative *cone* of $\mathcal{U}$ by $X^+ = \{x \in X \mid x \geq t\}$ and $X^- = \{x \in X \mid x \leq t\}$, respectively. Both cones are closed with respect to the monoidal operation $\circledast$; throughout the paper we will denote the negative and the positive cone operation of $\circledast$ by $\otimes$ and $\oplus$, respectively.
- A *chain* is a totally ordered poset. If X is linearly ordered, we speak about FL_e-*chains*. Call an FL_e-monoid *semi-linear* (or *representable*) if it can be represented as a subdirect product of FL_e-chains.
- Call $\mathcal{U}$ *conic* if every element of X is comparable with t, that is, if $X = X^+ \cup X^-$.
- Call a semi-linear, bounded, group-like FL_e-monoid a *SIU-algebra* ([23]), if the negative cone operation is the *dual* of the positive cone operation with respect to $'$, that is, if for $x, y \in X^-$, $x' \circledast y' = (x \circledast y)'$ holds.
- Call $\mathcal{U}$ *divisible* if for $x, y \in X$, $x \leq y$, there exists $z \in X$ such that $x = y \circledast z$. *BL-algebras* are divisible, semi-linear, bounded, integral FL_e-algebras with $f = \bot$ ([15]). *MV-algebras* are involutive BL-algebras ([9]). *Hoops* are divisible, commutative integral, residuated po-monoids.

[5]Commutative lattice ordered groups with $x \rightarrow_\circledast y := y \circledast x^{-1}$ are group-like.

- We call a group-like FL_e-monoid and also its monoidal operation $\circledast$ *absorbent-continuous* if for all $x \in X^-$, the absorbing set $\{z : z \circledast x = x\}$ of x has its least element.

Definition 1 (Ordinal sum construction) [2] Let $(I, \leq)$ be a totally ordered set. For each $i \in I$ let $\mathbf{A}_i = (A_i, \cdot_i, \rightarrow_i, 1)$ be a hoop such that for every $i \neq i$, $A_i \cap A_j = 1$. Then we can define the ordinal sum as the hoop $\bigoplus_{i\in I} \mathbf{A}_i = (\cup_{i\in I}, \cdot, \rightarrow, 1)$ where the operations $\cdot, \rightarrow$ are given by:

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if} \quad x, y \in A_i, \\ x & \text{if} \quad x \in A_i \setminus \{1\}, y \in A_j \quad \text{and} \quad i < j, \\ y & \text{if} \quad y \in A_i \setminus \{1\}, x \in A_j \quad \text{and} \quad i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} 1 & \text{if} \quad x \in A_i \setminus \{1\}, y \in A_j \quad \text{and} \quad i < j, \\ x \rightarrow_i y & \text{if} \quad x, y \in A_i, \\ y & \text{if} \quad y \in A_i, x \in A_j \quad \text{and} \quad i < j. \end{cases}$$

If in addition I has a minimum i_0 and $\mathbf{A}_{i_0}$ is a bounded hoop, then $\bigoplus_{i\in I} \mathbf{A}_i$ denotes the bounded hoop whose operations $\cdot, \rightarrow_i$ are defined as before, and whose bottom element is the minimum of $\mathbf{A}_{i_0}$. Each $\mathbf{A}_i$ is called a component of $\bigoplus_{i\in I} \mathbf{A}_i$.

Definition 2 (Twin-rotation construction – group-like case) [24] Let $(X_1, \leq)$ be a partially ordered set with top element t, and $(X_2, \leq)$ be a partially ordered set with bottom element t such that $(X, \leq)$, the connected ordinal sum of X_1 and X_2 (that is putting X_1 under X_2, and identifying the top of X_1 with the bottom of X_2) has an order reversing involution $'$ with fixed point t. Let $(X_1, \leq, \otimes, t)$ be a commutative, residuated monoid, and $(X_2, \leq, \oplus)$ be commutative, residuated semigroup. Let

$$\mathcal{U}_{\otimes}^{\oplus} = (X, \circledast, \leq, t, t)$$

where $\circledast$ is defined as follows:

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \setminus \{t\} \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in X_2, y \in X_1, \text{and } x \leq y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in X_1, y \in X_2, \text{and } x \leq y' \\ (y \rightarrow_{\otimes} x')' & \text{if } x \in X_2, y \in X_1, \text{and } x \not\leq y' \\ (x \rightarrow_{\otimes} y')' & \text{if } x \in X_1, y \in X_2, \text{and } x \not\leq y' \end{cases} . \tag{1}$$

Call $\circledast$ (resp. $\mathcal{U}_{\otimes}^{\oplus}$) the twin-rotation of $\otimes$ and $\oplus$ (resp. of the first and the second commutative residuated semigroup). It holds true that any conic, group-like FL_e-monoid can be represented as the twin-rotation of its negative cone and a slight modification of its positive cone, where its bottom element is reset to be zero of the multiplication [24].

Definition 3 [21] For a partially-ordered groupoid $(X, \leq, \circledast)$ over a complete lattice and for $x, y \in X \setminus \{\top\}$ we define

$$\begin{aligned} x \circledast_{co} y &= \inf\{x_1 \circledast y_1 \mid x_1 > x, y_1 > y\}, \\ x \circledast_Q y &= \inf\{x \circledast y_1 \mid y_1 > y\}. \end{aligned}$$

A major role in the proof of Theorem 5 is played by the following

Lemma 4 (Reflection Lemma) *Let $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ be a group-like FL_e-algebra over a complete, order-dense chain. For $\top \neq x, y \in X$,*

$$(x' \circledast y')' = x \circledast_{co} y = x \circledast_{Q} y.$$

Theorem 5 *$\mathcal{U} = (X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ is an absorbent-continuous group-like FL_e-algebra with involution $'$ over an order-dense and complete chain if and only if $\mathcal{U}$ is the twin-rotation of a BL-algebra and its de Morgan dual with respect to $'$, where the BL-algebra has components which are either cancellative[6] or MV-algebras over two elements,[7] and the BL-algebra cannot have two consecutive cancellative components.*

The result is very surprising since quite weak conditions characterise a quite specific class of algebras. [21, Examples 2 and 3] show that neither absorbent-continuity nor order density can be dropped from the conditions of Theorem 5. The lexicographic product of the totally-ordered additive groups of integers and real numbers forms an absorbent-continuous, densely-ordered, group-like FL_e-chain which is not complete, and is an example showing that completeness cannot be dropped either[8]. The operation $x \circledast y = \min\{0, x + y - 1\}$ over the real unit interval $[0, 1]$ is an example for an absorbert-continuous, not group-like involutive FL_e-algebra over a complete and order dense chain. Therefore, Theorem 5 is in its sharpest possible form.

References

[1] J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York-London, 1966.

[2] Agliano, P., Montagna, F.: Varieties of BL-algebras I: general properties, Journal of Pure and Applied Algebra, 181 vol. 2-3 (2003), 105–129.

[3] P. N. Ánh, L. Márki and P. Vámos, Divisibility theory in commutative rings: Bezout monoids, Trans. Amer. Math. Soc. 364 (2012), 3967-3992

[4] T. S. Blyth, M. F. Janowitz, Residuation Theory, Pergamon Press, 1972.

[5] B. Bosbach, Concerning bricks, Acta Math. Acad. Sci. Hungar. 38 (1981), 89-104

[6] B. Bosbach, Concerning semiclans, Arch. Math. 37 (1981), 316–324

[7] B. Bosbach, KomplementŁre Halbgruppen. Axiomatik und Arithmetik, Fund. Math. 64 (1969), 257–287

[8] B. Bosbach, Residuation groupoids and lattices, Studia Sci. Math. Hungar. 13 (1978), 433-451

[6] That is, those components are negative cones of totally ordered Abelian groups.
[7] Equivalently, Boole algebras over two elements.
[8] By courtesy of an anonymous referee.

[9] Cignoli, R., D'Ottaviano, I. M. L., D. Mundici: Algebraic Foundations of Many-Valued Reasoning, Kluwer, Dordrecht, 2000.

[10] Clifford, A. H.: Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.

[11] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, Oxford-London-New York-Paris, 1963.

[12] Galatos, N.: Minimal varieties of residuated lattices. Algebra Universalis, 52(2-3), (2005) 215–239.

[13] Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics vol. 151, Studies in Logic and the Foundations of Mathematics. Elsevier, (2007) 532 pp.

[14] G. K. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, Continuous Lattices and Domains, Encycl. of Math. and its Appl. 93, Cambridge U. Press, 2003.

[15] Hájek, P.: Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.

[16] O. Hölder, Die Axiome der Quantität und die Lehre vom Mass, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Classe, 53 (1901), 1–64

[17] Jenei, S.: Erratum "On the reflection invariance of residuated chains, Annals of Pure and Applied Logic 161 (2009) 220-227", Annals of Pure and Applied Logic, 161 (2010), 1603–1604.

[18] Jenei, S., On the geometry of associativity, Semigroup Forum, 74 vol. 3, (2007) 439–466.

[19] Jenei, S.: On the reflection invariance of residuated chains, Annals of Pure and Applied Logic, 161 (2009), 220–227

[20] Jenei, S.: On the structure of rotation-invariant semigroups, Archive for Mathematical Logic, 42(5), (2003), 489–514

[21] Jenei, S., Montagna, F.: A classification of certain group-like FL_e-chains, Synthese (2014). doi:10.1007/s11229-014-0409-2

[22] Jenei, S., Montagna, F.: Erratum to: A classification of certain group-like FLe-chains, Synthese (2015), doi: 10.1007/s11229-015-0778-1

[23] Jenei, S., Montagna, F.: Strongly involutive uninorm algebras, Journal of Logic and Computation, 23(3) (2013), 707–726

[24] Jenei, S., Ono, H.: On involutive FL_e-monoids, Archive for Mathematical Logic, 51 (2012), 719–738

[25] P. Jipsen, F. Montagna, Embedding theorems for classes of GBL-algebras, Journal of Pure and Applied Algebra, 214 vol. 9 (2010), 1559–1575

[26] P.T. Johnstone, Stone spaces, Cambridge University Press, Cambridge, 1982.

[27] P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math., 65 (1957), 117–143

[28] Ward, M. and R. P. Dilworth, Residuated lattices, Transactions of the American Mathematical Society 45: 335–354, 1939