A Pluralist Mathematical Practice

1. Introduction

"The Andréka-Németi group" is the honorary name given to some Hungarian logicians and scientists who have set up a programme to give the logical foundations of theories of physics. The core researchers in this area are Andéka, Németi, Madarász and Székely. In this paper I explain the methodology, show how it is different from other methodologies in science and point out in what sense it is pluralist, and in what ways it could become more pluralist. I also discuss the epistemological advantages and disadvantages of the methodology, by comparing it to others.

In the following sections, I shall explain three stages of the methodology. But it should be understood that one goes back-and-forth between the stages. None is ever completed.

2. The Methodology: Starting the Programme

I shall start by explaining how the methodology developed, but I shall accompany the explanation with more general philosophical remarks, as I contrast the work of the Andréka-Németi group to the more traditional methodology of physics. To help with the comparison, I'll dwell on the distinction between a 'law' of physics and an 'axiom' of logic.

The programme started with exploring 'the logical foundations' of special relativity. Here, 'the logical foundations' means a set of axioms written in a two-sorted first-order language. The Andréka-Némati group then developed the logical foundations of general relativity, and could show the formal logical relationship between general and special relativity. They are currently working on Newtonian mechanics[1] and quantum theory, and have ambitions to eventually develop the logical foundations of cosmology theory.

The methodology starts with the observed data of the physical theory. This stays fixed, since it is this that the members of the Andéka-Németi group want to understand. They then develop a logical language that can be used to represent the data. To develop the language, they ask the question: what is the theory about at least *prima facie*. The ontology of the special and general relativity theories is about bodies. There are several sorts. Photons, represented by the symbol 'Ph', inertial bodies, represented by the symbol 'Ib' and the more general notion of bodies, represented by the symbol 'B'. The latter is ambiguous between photons, other inertial bodies and accelerating bodies (the latter are only needed in general relativity).

They also ask the question: what does one do with these bodies in the science? The answer is that we make calculations about their spatio-temporal locations and trajectories 'from the point of view' of a body following a trajectory in space-time. So the other sort of symbol they have are quantitative 'Q'. These are then subdivided into: +, x, $\geq$, and so on. They then develop axioms that distinguish the ontological

[1] This work is undertaken primarily by Koen Lefever and Gergely Székely.

sorts[2] and capture the quantitative interrelations between the sorts of bodies. For example, one of the axioms distinguishes photons from other inertial bodies by fixing the speed of travel of photons as a constant from any 'point of view', and specifying that all other inertial bodies travel at a different speed. Quantities can also be sub-divided into time and space, or distance units. For ease of some calculations, the Andréka-Némati group fix a unit of measure. Also to accommodate transformations between reference frames of reference, we need quantities to measure angles.

To summarise, the logical language will have two general sorts: bodies, B, and quantities, Q. These are further sub-divided as and when this is convenient, necessary for accuracy of calculation or to appeal to our intuitions.

The Andréka-Németi group then work 'backwards' to find out what axioms could be used to derive the data as theorems. They also derive some of the textbook 'laws of physics' as theorems. Of course, the direction is also forwards: keeping the data fixed, they vary the language, the axioms and the formal methods of proof. The goal is to balance the logically simplest and more logically intuitive axioms against the simplest language against the most informative and convincing proofs. But they do not stop once they have found some logical axioms powerful enough to derive all the data. The axioms are not new laws of physics!

3. The Methodology: Continuing the Programme Within a Physical Theory

This is where the methodology departs from the usual methodology of physics and it has interesting philosophical implications. Once the Andéka-Németi group have one set of adequate axioms (from which they can derive all the data of the physical theory, say, special relativity), they then explore what happens if they change the axioms, by simplifying them or weakening them. There are two directions of what I shall henceforth call 'logical exploration'. One is inspired by the ideas of 'reverse mathematics', where we look for the weakest axioms (in some sense of 'weakest') to derive the data of the theory, the other direction of exploration is 'counter-factual'. The latter takes place when we ask: what would happen to the data if we were to modify an axiom thus, or if we were to omit this axiom all together? The logical exploration makes the explanations of the physical phenomena more *thorough*. We shall return to this point.

So, the Andréka-Németi group end up with a number of axiomatic systems each of which derives all, or some, of the observed (or observable) data as theorems of the formal system. So we have a plurality of formal axiomatic systems that *together* give the explanation of the physical data and the theory of physics.

No axiom is a physical law. Instead, it is an hypothesis. The logical relations between the formal theories is the subject of logical meta-exploration, and proofs and theorems are generated at this meta-level too. For example, the Andréka-Németi group might formally prove that two weaker axioms are equivalent to one stronger one.

This methodology is very different from the more traditional approach in physics where we look for the underlying laws of physical reality, and take them to

[2] These are 'definition'-type axioms.

be real truths about that reality. This tradition can be traced back, at least to Aristotle, who was looking for the underlying general principles that ordered kinematics, biology and even economics. Notice that I snuck in the word 'the'. The presupposition is that there exists a unique set of physical laws that can be conveyed in a natural or formal language. These laws fully, explicitly, clearly, without ambiguity or vagueness, without contradicting either each other or any other laws express the essence of the reality being investigated. These laws can then be used to make accurate predictions of all the phenomena in question. And the laws, in some sense, explain the science and the phenomena. In fact, some philosophers claim that an explanation in science requires such laws. These are strong pre-suppositions, and they explain the drive of present day scientists to discover these laws and work very hard at trying to find ways to 'unify' physics under one set of laws. Moreover, these presuppositions can be detected (at least) in chemistry and biology and have infected even some part of mathematics and economic theory! In fact, we might even say that, for some scientists, the search for the underlying laws *characterises* the scientific endeavour, so a questioning of the presuppositions is anathema to those practicing scientists. They cannot permit themselves to recognise *as science* any discipline or practice that departs from the presuppositions.

To draw out the difference, we might explore a little further the difference between an *axiom*, as it is used by the Andréka-Németi group, and a *law of physics*. A law of physics is supposed to be (some of): (1) everywhere true, (2) explanatory, (3) essential, (4) necessary (5) basic and (6) eternal. In the full paper, these will be discussed in turn.

In contrast, as it is used to mean by the Andréka-Németi group, an axiom is a hypothesis that one holds stable in an object-level theory. At the meta-level, it is possible to question and alter an axiom. The so called 'laws of physics' are then derived as theorems and are bench-marks that allow us to recognise the more traditional approach to physics, so that we can translate between the more traditional approach to physics and the Andréka-Németi approach.

We can make some important philosophical notes about 1-6 above. (1) In contrast to a law of physics, an axiom is not treated as true in an absolute sense, only true in a formal theory. (2) The Andéka-Németi axioms together with proofs of theorems that follow from the axioms are incomplete explanations. This is because they feature in 'explanations' that are supplemented by the further logical exploration.

For example, the explanation of a phenomenon does not end with a derivation of the phenomenon from some axioms. Instead, to complete the explanation, one looks to possibly other derivations, such as a combinatorial proof as opposed to a model theory proof. But this is not all. One also looks to alternative axioms to the ones used. For example, one might see if a weaker axiom is sufficient to derive the phenomenon in question, and if they find that an axiom is too weak, this negative information is also part of the explanation. It follows quickly that the axioms are also neither (3) essential nor (4) necessary (since they can be replaced by others that are different), although there will, presumably, be a weakest axiom capable of deriving the phenomena, when holding stable the other axioms of a given formal logical theory. But since we could change those axioms too, there might not be a unique 'weakest' set of axioms – mainly because what counts as 'weak' is not necessarily unique.

The comparison of (5) 'basicness' is also interesting. Understanding the laws of physics in the traditional approach requires us to develop some intuitions: physical intuitions. For example, we acquire an intuitive sense of what a photon is. In this sense, the laws of physics are basic. They cannot be reduced, or further analysed, beyond these basic intuitions common to the study of physics. But the Andréka-Németi group claim that this is just false. There is something more basic, and this is logic. It is when we use a logical language and make formal proofs, that we have reached the most basic level of explanation. We dispel the mysterious intuitions, and replace them with more solid logic. This is why the explanations are more thorough than those in the traditional approach. An obvious objection to the Andréka-Németi approach is that all we have done here is replace physical intuitions with logical intuitions. We shall return to this in the next section. As for (6) the eternal quality of the axioms, this is a very intriguing and interesting question, since they are not eternally true, *pace* the mathematical Platonists. They were written in a language that was developed at a time, moreover, how we understand them, interpret them and translate them into other formal language will change over time with our understanding of the surrounding mathematics. While this is an interesting question I shall probably not explore it further here.

<u>Continuing the Methodology: Extending it to Other Physical Theories.</u>

The programme continues. The methodology above can be used to give logical foundations for other physical theories. Note: 'foundations' here is not in the sense of one system of laws, but rather in the pluralist logician's sense of foundations: where there are several axiomatic systems that together found the theory in question. The advantage of using formal languages is that then the connections between the physical theories, can be made entirely logically explicit. For example, the relationship between special and general relativity is logically known and explicit. Lefever and Székely are working on giving the logical foundations of Newtonian mechanics. Again, the axioms are written in a first-order language which includes bodies and quantities. The calculations are a little different, since in Newtonian mechanics there is only one frame of reference. Once the logical foundations have been developed sufficiently to capture all of the observed data from axioms using formal proofs, and once several axiomatic theories have been developed that help us with a counter-axiomatic or limitative meta-understanding of Newtonian mechanics, then the logical relationship between the Andéka-Németi-Lefever-Székely Newtonian theories and the Andréka-Németi relativity theories can be made explicit. And so on for any other areas of physics founded in this manner.

<u>Pluralism in methodology</u>

I mentioned above that the methodology is logically pluralist. What does this mean? The methodology is pluralist in: (i) formal axiomatic systems, in (ii) ontology and in (iii) logic. Each of these terms will be explained, and more respects in which they methodology is pluralist will be added.

To be a pluralist in (i) formal axiomatic systems one has to hold that there is more than one axiomatic theory that is sufficient to derive or explain the theorems or data or phenomena being explored. Moreover, while one axiomatic system might be a 'favourite', there are others that are not equivalent and they are *plausible* or *admissible* in some sense. Some scientists looking at quantum mechanics, who are not part of the Andréka-Németi group are already pluralist in this sense. Such pluralist quantum theorists accept that there are very different theories of quantum mechanics that differ in ontology or logic and that are each plausible. This does not preclude them from having a favourite theory, or from working exclusively in one theory of quantum mechanics. Nevertheless, other theories might be informative or interesting, and it follows that the laws, or the ontology, or the metaphysics, or the logic of the theories will be different. They might hope, or think, along with the presuppositions of the traditional approach, that eventually, once enough research is done, one set of laws, logic, metaphysics etc. will 'win out'. This is presently a typical attitude, but it is hardly decisive, since the Andréka-Németi group's methodology provides an alternative philosophical approach to physics that does not hold the same presuppositions.

The Andréka-Németi group are pluralist in formal axiomatic system, since they have several axiomatic systems, that together, provide the explanation for the phenomena of special and general relativity. But they are more pluralist than in this sense, since they are also interested in the question of when it is that a hypothesised axiom *fails* to be strong enough to derive the desired data. Members of the Andréka-Németi group find that the negative information adds to the understanding of the general physical theory. The negative information is a sort of counter-factual exploration inspired by the logician's notion of a 'limitative result' and the approach of reverse mathematics. But it is not quite counter-*factual*. Rather, it is more counter-*axiomatic*, since the information is derived using different axioms.

To be a pluralist in (ii) ontology, one has to think that there are different ontologies. This sounds like a radical claim, since some people think of this as amounting to the claim that there is not one fundamental 'reality'. And yet, it might not be as radical as one thinks. There are degrees of ontological commitment. Let us start with the easiest.

When our science is going well, our formal (or formal as possible) theory represents the ontology perfectly up to the requirements of capturing existing known data, predicting future data and even calculating hypothetical data. In particular, in the Andréka-Németi case, the formal language has sort variables varying over bodies and quantities. How we understand those bodies and quantities will change when we change theories. For the purposes of discussion here, we individuate theories by their axioms, definitions, language and allowed proof methods. For example, we could introduce superluminal particles (bodies travelling faster than the speed of light) through an axiom that is consistent with axioms of another theory to make a new theory. The two theories differ in ontology in the sense that one theory has a sort of body that is not present in the other theory. This pluralism in ontology is not radical, since one theory is really just a sub-theory of the other, and we could think that the superluminal particles were 'in' the other theory, we had just not mentioned them in the language and axioms (yet).

However, the Andréka-Németi group's pluralism in ontology is not quite so banal. For, the purpose of their giving logical foundations for special and general relativity is to do away with having to mysteriously acquire intuitions concerning the ontology. So, instead of developing a warm feeling about photons and so on, we learn about 'them' through the axioms and making proofs, derivations and transformations. We then are meant to understand 'photons' better, since more thoroughly, and in the light of having negative information about 'them'; where *negative information* here means that we know what phenomena we can deduce when all axioms about 'those entities' are weakened. There are two ways of taking these claims. One is a weak way. Another is a strong way.

The weak way is to deny that this is a pluralism in *ontology*, and that instead it is a pluralism in *epistemology*: where we understand 'the' entities differently through the different theories, but there is still one underlying reality. Take off the single quotation marks in the above paragraph, and it reads with this this weak interpretation.

For the strong interpretation take the single quotation marks seriously. Re-read the paragraph, this time with every word in single quotation marks as a parameter for admissible entities. That is, try to imagine that there are several things that photons could be, the class of which is individuated differently in each theory, and there might be several things in that class. When we do this exercise we discover that 'photon' is a placeholder for a number of possible entities. This is to take very seriously the idea that theories do not so much *represent the* ontology, but rather, they determine the parameters for the ontology in a way that is consistent with the observed data. This is a much more 'top-down' or logic-inspired way of looking at the theories and it allows for a strong ontological relativism. There is nothing to force the strong reading over the weak reading. Both are consistent with the Andréka-Németi approach.

(iii) Pluralism in logic: members of the Andréka-Németi group are pluralists in logic, but in a fairly conservative sense. Again, there are degrees of logical pluralism. A logical pluralist thinks that there are necessary and sufficient conditions which have to be met in order for a formal system to claim to be a formal 'logic', and there are several 'logics', i.e., several formal systems that satisfy the conditions. The result is that some formulas are logical truths in one logic and not in another. The different degrees depend on the particular necessary and sufficient conditions.

There is a straightforward and 'low degree' sense in which this applies to the work of the Andréka-Németi group. Their logical reasoning is classical, and their language is first-order. Nevertheless, they have several theories. The several theories have different axioms, in some theories some formula will be derived and in another theory a formula that precludes the first will be derived. In particular, this is the case if we think of what I have been calling 'logical exploration'. For example, we might find out that clocks do not slow down under circumstances (sets of formulas) X in theory T, but they do under circumstances X in theory T*. In other words, if we were to put the axioms of the two theories together, we might well find that we have a contradictory theory: one in which a formula and its negation can both be derived. If the Andréka-Németi group are very careful, and all of the theories are nested in the right way, and we are precluded from mixing arbitrary theories together, then contradictions like this should not arise as a result of calculation of some observable data (of course *reductio* proofs are allowed, since assumptions made for the *reductio* are discharged). That is, there are ways of preserving consistency, because we might be able to derive a formula in one theory but not in the other.

The next, less obvious degree of logical pluralism is to allow more logical operators into the language and this comes, of course, with modifications to the axioms. Some members of the Andréka-Németi group are logical pluralists to this degree. Molnár has added modal operators for reasoning over other possible worlds, adding significantly to our 'counter-axiomatic' understanding of the physical theories. The Andréka-Németi group give necessary but not sufficient conditions for something to count as a logic. Therefore, it is possible to add other operators, should this prove interesting.

However, they could be more radical and weaken the necessary conditions to include non-classical logics. They are not quite ready to do this, and it might be a mistake politically, since they are already departing from traditional physics in two ways, one is to give up the presuppositions about laws discussed above. The other is to have quite an advanced understanding of logic – technically and philosophically. To ask for even more understanding of logic to include deploying non-classical logics might be too much. However, entertaining the possibility, raises a question I left pending above. The question concerned logical intuitions.

Put starkly, we might deny that the Andréka-Németi group are providing a more 'basic' theory at all, and suggest that instead, all they are doing is replacing physical intuitions with logical intuitions. This criticism will be answered in the paper in a very interesting way using a *reductio* argument to show that we probably can't make much sense of our physical intuitions!

This criticism would fail if we thought that there is just one logic – in the sense that there is one and only one norm for reasoning, and everyone tries to adhere to it, and recognises corrections (ultimately only) with regard to that norm. The norm of reasoning is represented by a formal system, say, classical first-order logic. If we follow this train of thought, then 'logical intuitions' are hardly suspect in the same way as physical intuitions are (that are over-turned with paradigm shifts in science). The thought goes that logical intuitions, in contrast to scientific intuitions, come from our shared sense of normativity in reasoning. Unfortunately, things are not so simple in the light of the logical pluralism described above, even at the lowest degree of logical pluralism.

Since they are logical pluralists, members of the Andréka-Németi group admit different sets of logical intuitions. One problem with responding to the accusation that we are replacing physical with logical intuitions is to know what we mean by logical intuitions. We might think that physical intuitions are fairly straightforward. There

if logical intuitions are distinguished by their respective theories. For example, the sort Ib will correspond to a different logical 'intuition' when implicitly or explicitly defined by one set of axioms or another.

There are logical monists who adhere to quite different formal logical systems as representative of reasoning. We might explain this by saying that they have different logical intuitions. Some logical monists think that the 'law of non-contradiction' is true, and is a law of reasoning, that we cannot do reasoning in the face of a contradiction. Others think that there is no good argument for never recognising contradiction, and they supply formal systems of logic that allow us to reason rigorously in the light of some contradictions, without collapsing into triviality. So, as a matter of present-day fact there is no evidence that there is only one norm for reasoning or one formal system that represents that reasoning, therefore, given the empirical evidence of the current practice of developing formal logical systems, we cannot just be replacing physical intuitions with logical intuitions, since there is there are a number of quite different logical intuitions. One set of logicians might, ultimately be right, but on present evidence we cannot tell which ones are right, or even if there is one unique norm. This argument will be further developed in the presentation.

Conclusion

The methodology developed by the Andréka-Németi group is not for everyone. It requires logical sophistication, and is best suited to those who gain understanding through logic, not through intuitions of physical reality. To use the methodology, one has to give up some of the presuppositions that are made when we are traditionally presented with a scientific theory. I have illustrated some of these differences by discussion the distinction between laws and axioms. As a method for understanding science, the advantages come from asking logician's questions about the theory and data. They include an understanding of the logical limitations of the theory and the ability to make predictions in the form of, for example, showing that some phenomena such as superluminal particles, is *logically consistent with* the theories of relativity.

Bibliography

Andéka, H., J.X. Madarász, I. Nemeti. (2002). 'On the logical structure of relativity theories', E-book.

Andéka, Hajnal, Judit X. Madarasz, István Németi and Gergely Székely. (2011). A Logic Road from Special Relativity to General Relativity. *Synthese*. Special issue: Logic Meets Physics. I.3, 633 – 649.

Curd, Martin and J.A. Cover. (1998). Philosophy of Science. W.W. Norton & Company

Friend, Michèle and Andrea Pedeferri. (2011) An Analysis of the Notion of Rigour in Mathematical Proof. L&PS – *Logic and Philosophy of Science*. Vol. IX, No.1.

Friend, Michèle. (2014). *Pluralism in Mathematics; A New Philosophical Position*. Logic, Epistemology and the Unity of Science. Springer.

Friend, Michèle and Molinini, Daniele. (Forthcoming *Philosophia Mathematica*). 'When Mathematics Explains a Scientific Theory'.

Madarász, Judit and Székely, Gergely. (2013). Special Relativity over the Field of Rational Numbers. *International Journal of Theoretical Physics.* 52.5 (1706 – 1708).

Németi, Péter, Székely, Gergely. (2012) Existence of Faster then Light Signals Implies Hypercomputation already in Special Relativity. Ch. 10 of *How the World Computes*, Lecture Notes in Computer Science, Vol. 7318, pp. 528 – 538.

Szekely, Gergely. (2009). First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers. PhD Thesis. http://www.renyi.hu/~turms/phd.pdf

Székely, Gergely. (2012a), 'The Existence of Superluminal Particles is Consistent with the Kinematics of Einstein's Special Theory of Relativity', Manuscript, arXiv:1202.5790v1.

Székely, Gergely. (2012b). 'What properties of numbers are needed to model accelerated observers in relativity?' Manuscript. http://arxiv.org/1210.0101

Székely, Gergely (2013). 'The Existence of Superluminal Particles is Consistent with Relativistic Dynamics'. Manuscript. http://arxiv.org/abs/1303.3099

Wright, Crispin. (1992), *Truth and Objectivity*, Cambridge, Massachusetts: Harvard University Press.