Definitions and Contextualism for Topologies on the Space of Spacetimes

Samuel C. Fletcher*

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1 Introduction

There is an intuitive idea that some spacetime models are "similar" to others in some relevant respect or other. A natural way to make this notion precise is to put a topology over spacetimes—that is, over the smooth Lorentz metrics $L(M)$ of a fixed[1] manifold M—that respects the aspects that one wishes to capture. Besides specifying which sequences of spacetimes converge and which parameterized families of spacetimes vary continuously, a topology determines which properties of spacetimes are stable and generic.[2] It may also play a role in determining when a perturbation of a spacetime metric is to be considered small.

Of course, there are many topologies from which to choose. The following sections will clarify the status of a few of the more common topologies—the point-open, compact-open, and open topologies—elucidating some of their properties and relationships, and showing that some alternative definitions are in fact equivalent.

Through this investigation, it will become clear that different topologies are natural choices for different questions. Thus, through some generalizations of propositions of Fletcher (2015), I argue that there is good reason to believe that there can be no canonical topology over spacetimes—no topology that can capture at once all of the roles listed above. Thus, in a sense, the many demands of this intuition of "similarity" pull in different, sometimes incompatible directions. It thus seems best to accept a kind of methodological contextualism, where the best choice of topology is the one that captures the properties relevant to the research question at hand that "similar" spacetimes should share.

2 Topologies on the Space of Spacetimes

There are two ideas on how to express the topologies on $L(M)$ considered in this paper. The first uses the fiber bundle formulation of $L(M)$, and in particular jet bundles thereof.

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[1]One might of course wish to compare spacetimes whose underlying manifolds are not identical or even homeomorphic, but for now I will set this possibility aside.

[2]A property P of a spacetime g is stable when there is an open neighborhood of g, each element of which has P. P is generic within a subset $S \subseteq L(M)$ when it holds on each element of an open dense subset of S.

Let $\pi^0_2 : T^0_2 M \to M$ be the canonical projection associated with the bundle $T^0_2 M$ and note that there is a natural equivalence between the set of $(0, 2)$-tensor fields on M, $\Gamma^0_2(M)$, which include the Lorentz metrics $L(M)$, and a certain set of cross-sections $\hat{\Gamma} : M \to T^0_2(M)$, which, by definition, satisfy $\pi^0_2 \circ \hat{\Gamma} = I$, the identity. Denote these sections by $\hat{\Gamma}^0_2(M)$, and the corresponding sections for Lorentz metrics as $\hat{L}(M)$. The k-jet of a cross-section $\hat{\Gamma}$ at $p \in M$, denoted $j^k\hat{\Gamma}(p)$, is the equivalence class of sections whose partial derivatives in an arbitrary coordinate system at p are equal to those of Γ up to order k. The k-jets of sections themselves form a bundle over M, whose total space will be denoted $J^k(M, \hat{\Gamma}^0_2)$. The open sets of this bundle can be written in the form of ordered pairs—i.e., they constitute a product topology—where the first component is homeomorphic to an open set from the manifold topology on M and the second is homeomorphic to an open set of tuples whose values take on the possible values in the typical fiber $(\pi^0_2)^{-1}[p]$ and their partial derivatives to order k. The latter is a finite-dimensional real vector space, for which there is a unique reasonable (i.e., Hausdorff) topology corresponding to the topology induced by the Euclidean norm. (See, for example, (Schafer, 1999, Theorem 3.2, p. 20).) Now consider some collection of sets $\mathcal{S}$ that cover M (i.e., $\bigcup \mathcal{S} = M$). For any open set $U \in J^k(M, \hat{\Gamma}^0_2)$ and $S \in \mathcal{S}$, define $O^k(U, S) = \{g \in L(M) : j^k\hat{g}[S] \subseteq U\}$ as the set of metrics g whose corresponding k-jet cross-sections $j^k\hat{g}[S]$ are in U. These $O^k(U, S)$ form a basis for a topology.

The second idea for topologies on $L(M)$ considered in this paper, is that one can divide the task of measuring the similarity of a pair spacetimes into two parts: first, that of encoding their relevant differences at each point of M into a real number in some systematic way, and second, evaluating the variability of the resulting scalar field in some way over regions of M. The differences between the topologies considered arise ultimately from different choices of how to implement these two tasks. In fact, most of these differences arise in the second part of this task.

To show how this works in more detail, some definitions are needed. A *seminorm* on a vector space X is a subadditive and homogeneous function $|\cdot| : X \to \mathbb{R}$, i.e., one such that, for all $x, y \in X$ and scalars c, $|x + y| \leq |x| + |y|$ and $|cx| = |c| \cdot |x|$. A family of seminorms $\mathcal{N}$ on X is called *separating* when for each nonzero $x \in X$, there is some $|\cdot| \in \mathcal{N}$ such that $|x| \neq 0$.

Since a space of (r, s)-tensors, such that those in each fiber of the tensor bundle $T^r_s M \to M$, is a vector space, one may in particular define seminorms that vary continuously (smoothly) across M, in the sense that for any continuous (smooth) (r, r)-tensor field K,[3] $|K|$ is a continuous (smooth) real scalar field on M. Such a varying norm will be called a *fiber norm* for the (r, r)-tensor fields.

A family of fiber norms for (r, s)-tensor fields for *each* $r, s \geq 0$ can be induced from a Riemannian metric h. (Such a metric always exists since M is assumed to be Hausdorff and paracompact.) This is the first part of the task mentioned above. Define the *h-fiber norm* $|\cdot|_h$ of any (r, r)-tensor field K as the fiber norm

$$|K|_h = \begin{cases} |K|, & \text{if } r = s = 0, \\ |K^{a_1\cdots a_r}_{b_1\cdots b_s} K^{c_1\cdots c_r}_{d_1\cdots d_s} h_{a_1c_1} \cdots h_{a_rc_r} h^{b_1d_1} \cdots h^{b_sd_s}|^{1/2}, & \text{otherwise.} \end{cases} \tag{1}$$

Note that the h-fiber norm of a scalar is just the absolute value of that scalar, hence independent of the choice of h.

[3]For explicit tensor calculations I use the abstract index notation, but otherwise I drop tensor indices to reduce notational clutter.

The second part of the task then involves defining bases for the topology using sets of the form

$$B^k(g, \tilde{\epsilon}; h, S) = \{g' \in L(M) : (|g - g'|_h)_{|S} < (\tilde{\epsilon})_{|S}, \ldots, (|\nabla^{(k)}(g - g')|_h)_{|S} < (\tilde{\epsilon})_{|S}\}, \quad (2)$$

varying over all $g \in L(M)$, positive scalar fields $\tilde{\epsilon}$, Riemannian h, and sets $S \in \mathcal{S}$ such that $\bigcup \mathcal{S} = M$. I will also use the notation $d_h(g, g') = |g - g'|_h$ in what follows.

2.1 The Open Topologies

The C^k *open* topologies control similarity of spacetimes across the whole of the underlying manifold. For simplicity I consider here only the C^0 case; similar analysis and results apply to the arbitrary C^k case. The C^0 *open* topology is perhaps best known in the context of Hawking's theorem (Hawking and Ellis, 1973, Prop. 6.4.9, p. 198), that a spacetime is stably causal iff it admits of a global time function. But different definitions have been advanced in the literature. For example, (Geroch, 1971, p. 71) (who calls it the "fine" topology) writes that, in this topology, "a neighborhood of a metric gab consists of metrics which lie within a certain range of g_{ab} at each point, where this 'range' varies continuously but otherwise arbitrarily over the space-time." What does it mean for a metric to lie in a certain range of another? Hawking (1971) makes it explicit that these ranges are to be measured by some positive-definite metric h_{ab} on the manifold M. So this would suggest that we could define the open neighborhoods B of g_{ab} by

$$B(g, \tilde{\epsilon}; h) = \{g' \in L(M) : |g - g'|_h < \tilde{\epsilon}\}, \quad (3)$$

where $\tilde{\epsilon}$ is some positive continuous scalar field on M.[4] (Indeed, Lerner (1973) uses this definition in an appendix.) However, in a footnote to the above passage Geroch instead defines

$$B_{max}(g, \epsilon; h) = \{g' \in L(M) : \max_M |g - g'|_h^2 < \epsilon\}, \quad (4)$$

where (allowing for the slight abuse of notation) ϵ is some positive constant. Earlier, (Geroch, 1970, p. 279) effectively defines

$$B_{sup}(g, \epsilon; h) = \{g' \in L(M) : \sup_M |g - g'|_h^2 < \epsilon\}. \quad (5)$$

The sets of each of these, ranging over all possible choices of g, $\tilde{\epsilon}$ or ϵ, and h, serve as *bases* for three topologies. (They each generate a topology over arbitrary union.) Although it may not be obvious, they in fact generate the same topology. To show why, we will use the following lemma:

Lemma 1. *Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, on a set X. Then $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if and only if for every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$, there is some $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

For a proof, see, e.g., (Munkres, 2000, Lemma 13.3, p. 81). Let $\mathcal{T}$, $\mathcal{T}_{max}$, and $\mathcal{T}_{sup}$ be the topologies defined by equations 3, 4, and 5, respectively, for a fixed manifold M.

Proposition 1. $\mathcal{T} = \mathcal{T}_{max} = \mathcal{T}_{sup}$.

[4] The choice of two copies of h ensures that the "distance" thereby defined between g and g' is zero iff $g = g'$, i.e., that the resulting topology is Hausdorff.

Proof. We use the above lemma three times. First, pick an arbitrary $g \in B_{sup}(g, \epsilon; h)$, and consider $B(g, \tilde{\epsilon}; h)$, where $\sup_M \tilde{\epsilon} < \epsilon$. For any $g' \in B(g, \tilde{\epsilon}; h)$, we have $h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) < \tilde{\epsilon}$, hence $\sup_M h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) \leq \sup_M \tilde{\epsilon} < \epsilon$. Therefore $g' \in B_{sup}(g, \epsilon; h)$ by definition, and $\mathcal{T}_{sup} \subseteq \mathcal{T}$ by the lemma.

In a similar fashion, pick an arbitrary $g \in B(g, \tilde{\epsilon}; h)$, and consider any $g' \in B_{max}(g, 1; h/\sqrt{\tilde{\epsilon}})$. We have $\max_M \frac{h_{am}}{\sqrt{\tilde{\epsilon}}} \frac{h_{bn}}{\sqrt{\tilde{\epsilon}}}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) < 1$, hence in particular $h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) < \tilde{\epsilon}$. Therefore $g' \in B(g, \tilde{\epsilon}; h)$ by definition, and $\mathcal{T} \subseteq \mathcal{T}_{max}$ by the lemma.

Lastly, pick an arbitrary $g \in B_{max}(g, \epsilon; h)$, and consider any $g' \in B_{sup}(g, 1; h/\sqrt{\epsilon\rho})$, where ρ is a continuous scalar field on M satisfying the following two conditions: $0 < \rho \leq 1$ everywhere, and the upper level sets $L^+(c) = \{p \in M : \rho_{|p} \geq c\}$ are compact for each $c > 0$. (The latter condition, along with the positivity of ρ, capture a sense in which ρ may be said to "vanish at infinity " if M is non-compact.) To simplify the notation, put $\gamma = \frac{h_{am}}{\sqrt{\epsilon\rho}} \frac{h_{bn}}{\sqrt{\epsilon\rho}}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn})$. We then have $sup_M \rho < 1$, hence in particular

$$\rho\gamma = \frac{h_{am}}{\sqrt{\epsilon}} \frac{h_{bn}}{\sqrt{\epsilon}}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) < \rho \leq \tilde{1},$$

where $\tilde{1}$ is a constant unit scalar field. To show that $g' \in B_{max}(g, \epsilon; h)$, hence $\mathcal{T}_{max} \subseteq \mathcal{T}_{sup}$, it suffices to show that $\max_M \rho\gamma$ exists. This is trivial if M is compact, so consider the non-compact case. Since $\rho\gamma$ is bounded, $\sup_M \rho\gamma$ must exist and be strictly positive (since otherwise $\gamma = 0$ everywhere and the maximum exists trivially). Thus, for any $\delta > 0$, consider $O_\delta = \{p \in M : (\sup_M \rho\gamma) - (\rho\gamma)_{|p} < \delta\}$, the set of points of M on which $\rho\gamma$ is within δ of its supremum. Choosing $\delta < \sup_M \rho\gamma$, we have that $p \in O_\delta$ only if $\sup_M -\delta < (\rho\gamma)_{|p} < \rho_{|p}$, hence $O_\delta \subset L^+(\sup_M \rho\gamma - \delta)$. But this means that the closure of O_δ is compact, hence by continuity $\rho\gamma$ must attain a maximum thereupon. Since points p on the boundary of O_δ satisfy $(\sup_M \rho\gamma) - (\rho\gamma)_{|p} = \delta$, this maximum must lie on the interior. Continuity again implies at last that $\sup_M \rho\gamma$ attains at this maximum. □

The fiber bundle approach to defining the open topology is taken by Hawking (1969), Lerner (1973), and (Hawking and Ellis, 1973, p. 198). Here, one takes as a subbasis for the topology $\mathcal{T}_O$ the sets of the form $O(U) = \{g \in L(M) : \hat{g}[M] \subseteq U\}$ for open sets U of $T^0_2 M$. In fact, as suggested, this topology is also identical to the ones discussed above.

To see why, define $\mathcal{T}(h)$ as the topology generated from equation 3 but with a single choice of positive definite metric h. That is, the open sets of $\mathcal{T}(h)$ are formed by taking arbitrary unions of sets of the form given in equation 3, ranging over just Lorentz metrics g and positive continuous fields $\tilde{\epsilon}$.

Proposition 2. *For any positive definite h, $\mathcal{T}(h) = \mathcal{T}_O$.*

Proof. Let $\hat{L}(M)$ denote the Lorentz metrics on M taken as cross-sections $\hat{g}$ of the bundle $T^0_2(M)$ with associated projection π, and define the scalar field $d_h(\hat{g}, \hat{g}') = h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn})$. Now let $B(g, \tilde{\epsilon}; h) = \{g' \in L(m) : d_h(\hat{g}, \hat{g}') < \tilde{\epsilon}\}$ be given, and consider the mapping $\Delta : \hat{L}(M) \to \mathbb{R}$ defined by $\hat{g}' \mapsto (\tilde{\epsilon} - d_h(\hat{g}, \hat{g}'))_{|\pi(\hat{g}')}$, which gives the (signed) difference between the field $\tilde{\epsilon}$ and the "distance" between g and g' at any point of M. Since Δ is continuous, the set $U = \Delta^{-1}(0, \infty)$ is open, hence by construction $O(U) = B(g, \tilde{\epsilon}; h)$. Because $B(g, \tilde{\epsilon}; h)$ was arbitrary, we have that $\mathcal{T}(h) \subseteq \mathcal{T}_O$.

For the converse, let W be an arbitrary open neighborhood of g in $\mathcal{T}_O$ and let V be any open set of $T^0_2(M)$ such that $g \in O(V) \subseteq W$. Now define the scalar field $m : M \to \mathbb{R} \cup \{\infty\}$

so that for any $p \in M$,

$$m_{|p} = \begin{cases} \inf\{d_h(\hat{g}, \hat{g}')_{|p} : \hat{g}' \in T^0_{(2)}(M) - V\}, & \pi^{-1}(p) \nsubseteq V, \\ \infty, & \pi^{-1}(p) \subseteq V. \end{cases}$$

The field m gives at each point the greatest lower bound of the "distance" between g and the set of g' whose corresponding cross-sections $\hat{g}'$ are outside of V. Now, since M is paracompact and Hausdorff, it has a locally finite covering by compact sets.[5] (See, e.g., (Munkres, 2000, Lemma 41.6, p. 258).) The field m is bounded from below by a positive constant on each such compact set, so there exists a continuous field $\tilde{\epsilon}$ such that $0 < \tilde{\epsilon} \leq m$ everywhere. (Apply the partition of unity argument of (Munkres, 2000, Theorem 41.8, p. 259).) So for any $g' \in B(g, \tilde{\epsilon}; h)$, we must have $d_h(\hat{g}, \hat{g}') < m$, which means that $\hat{g}' \in V$. But then $g' \in O(V)$, hence $B(g, \tilde{\epsilon}; h) \subseteq O(V)$ and by the lemma, $\mathcal{T}_O \subseteq \mathcal{T}(h)$. □

One might have thought that $\mathcal{T}$, despite being well-defined and having a conveniently expressible basis, was somewhat ad hoc. Why should a positive definite metric h on M—an object mathematically foreign to the concerns of relativity—matter in how we decide if two spacetimes are close, even if we do not relativize to a particular choice of h? The above proposition shows that this concern is in fact misplaced, because the basis of $\mathcal{T}$ is indeed just a convenient way to express the topology that arises naturally from the fiber bundle structure.

Now, in the proof of the first proposition, I utilized a certain redundancy in the definitions of the bases between the choice of h and the choice of $\tilde{\epsilon}$. The above proposition implies that in fact a single choice suffices:

Corollary 1. *For any positive definite h and h', $\mathcal{T}(h) = \mathcal{T}(h') = \mathcal{T}$.*

An analogous proposition holds for the alternative definitions of the open topology only when M is compact. Let $\mathcal{T}_{sup}(h)$ and $\mathcal{T}_{max}(h)$ be the topologies generated by letting equations 4 and 5 range over ϵ and g only.

Proposition 3. *If M is compact, then for any two positive definite metrics h and h', $\mathcal{T}_{max}(h) = \mathcal{T}(h')_{max} = \mathcal{T}_{sup}(h') = \mathcal{T}_{sup}(h)$.*

Proof. Clearly $\mathcal{T}_{max}(h) = \mathcal{T}_{sup}(h)$ when M is compact, so it suffices to prove the first equality. Let $B_{max}(g, \epsilon; h)$ and some positive definite metric h' be given. For brevity, define $d_h(g, g') = h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn})$ and

$$\delta(g, g') = \epsilon \frac{\max_M d_{h'}(g, g')}{\max_M d_h(g, g')}$$

for all $g \neq g'$. Note that δ is well-defined since, by the compactness of M, the numerator is always finite, and the denominator would vanish iff $g = g'$. I claim that $\epsilon' = \inf\{\delta(g, g') : g' \in B_{max}(g, \epsilon; h) - \{g\}\}$ is nonzero. Suppose otherwise. Noting that δ is a continuous function in the topology $\mathcal{T}_{max}(h)$,[6] it either achieves the infimum in the interior of $B_{max}(g, \epsilon; h) - \{g\}$ or on its boundary. But on the interior, the denominator of

[5] A collection of subsets of a topological space is *locally finite* when every point in the union of the collection has a neighborhood intersecting only finitely many members of that collection.

[6] The precise choice of topology does not matter as long as δ is continuous and, as defined below, $\overset{n}{g} \to g$. In particular, the value of ϵ' does not depend on this choice.

δ is positive and bounded while the numerator would achieve zero iff $g = g'$. The boundary, on the other hand, is given by points of the form $\{g' \in L(M) : \max_M d_h(g, g') = \epsilon\}$ and the point g. In the former case, $\delta(g, g') = d_{h'}(g, g') \neq 0$, so we must have that the infimum occurs at g. This entails in particular that $\delta(g, \overset{n}{g}) \to 0$ for any sequence $\overset{n}{g} \to g$ in the topology $\mathcal{T}_{max}(h)$. So consider the sequence

$$\overset{n}{g}_{ab} = \left(1 + \sqrt{\frac{c_n}{h^{am}h^{bn}g_{ab}g_{mn}}}\right) g_{ab},$$

where the c_n are positive constants satisfying $c_n \to 0$. Note that $\overset{n}{g} \to g$ in the topology $\mathcal{T}_{max}(h)$, while

$$\delta(g, \overset{n}{g}) = \epsilon \max_M \frac{h'^{am}h'^{bn}g_{ab}g_{mn}}{h^{am}h^{bn}g_{ab}g_{mn}}$$

is a nonzero constant independent of n, which is a contradiction. Therefore $\epsilon' > 0$.

Now consider any $g' \in B_{max}(g, \epsilon'; h')$. By definition we have

$$\max_M d_{h'}(g, g') < \epsilon' \leq \epsilon \frac{\max_M d_{h'}(g, g')}{\max_M d_h(g, g')},$$

which implies that $\max_M d_h(g, g') < \epsilon$. Hence $B_{max}(g, \epsilon'; h) \subseteq B_{max}(g, \epsilon; h)$, and by the lemma, $\mathcal{T}_{max}(h) \subseteq \mathcal{T}_{max}(h')$. A similar argument proves the converse. □

Proposition 4. *If M is non-compact then for every positive definite metric h, there is a positive definite metric h' such that $\mathcal{T}_{sup}(h)$ is incomparable with $\mathcal{T}_{sup}(h')$, and $\mathcal{T}_{max}(h)$ is incomparable with $\mathcal{T}_{max}(h')$.*

Proof. Let h be given. Put $h' = \sqrt{\rho}h$, where ρ is any continuous positive scalar field on M such that $\inf_M \rho = 0$ but $\sup_M \rho$ does not exist (i.e., is infinite). Now suppose, for the sake of contradiction, that for any $\epsilon > 0$ there is some $\epsilon' > 0$ for which $B_{sup}(g, \epsilon'; h') \subseteq B_{sup}(g, \epsilon; h)$. Consider

$$g'_{ab} = \left(1 + \sqrt{\frac{c'}{h'^{am}h'^{bn}g_{ab}g_{mn}}}\right) g_{ab} \in B_{sup}(g, \epsilon'; h'),$$

where $0 < c' < \epsilon'$, and note that $h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn}) = c'/\rho$, whose supremum does not exist. Hence $B_{sup}(g, \epsilon'; h') \nsubseteq B_{sup}(g, \epsilon; h)$, and by the lemma, $\mathcal{T}_{sup}(h) \nsubseteq \mathcal{T}_{sup}(h')$.

Conversely, suppose that for any $\epsilon' > 0$ there is some $\epsilon > 0$ for which $B_{sup}(g, \epsilon; h) \subseteq B_{sup}(g, \epsilon'; h')$. Consider

$$g''_{ab} = \left(1 + \sqrt{\frac{c}{h^{am}h^{bn}g_{ab}g_{mn}}}\right) g_{ab} \in B_{sup}(g, \epsilon; h),$$

where $0 < c < \epsilon$, and note that $h'^{am}h'^{bn}(g_{ab} - g''_{ab})(g_{mn} - g''_{mn}) = c\rho$, whose supremum does not exist. Hence $B_{sup}(g, \epsilon; h) \nsubseteq B_{sup}(g, \epsilon'; h')$, and by the lemma, $\mathcal{T}_{sup}(h') \nsubseteq \mathcal{T}_{sup}(h)$.

Therefore we may conclude that $\mathcal{T}_{sup}(h)$ is incomparable with $\mathcal{T}_{sup}(h')$. A similar proof shows the same for $\mathcal{T}_{max}(h)$ and $\mathcal{T}_{max}(h')$. □

Thus we see that differences between, on the one hand, the neighborhoods $B(g, \tilde{\epsilon}; h)$, and on the other, the neighborhoods $B_{sup}(g, \epsilon; h)$ and $B_{max}(g, \epsilon; h)$, arise only when M is non-compact. This makes sense. On compact manifolds, the field $h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} -$

g'_{mn}) is always bounded, so taking its maximum (or, equivalently, its supremum) does not impose any further constraint. By contrast, on non-compact manifolds one must consider the behavior "at infinity."

This is one way of beginning the elaboration of a comment by (Hawking, 1971, p. 393) that, in defining topologies over spacetimes, one must concern oneself especially with the nature of "the regions on which the metrics are required to be near. This is really a question of how the metrics behave near the edge of the manifold, i.e. near infinity." In our further study of topologies over spacetimes, especially the point-open and compact-open topologies, this question will arise more acutely.

2.2 Compact-Open Topologies

In the previous section we concerned ourselves with a topology that exerts global control when adjudicating whether two spacetimes are close. A weaker condition is to control how they differ on compact sets. Consider the following two neighborhood systems:

$$B_C(g, \tilde{\epsilon}; h, C) = \{g' : (|g - g'|_h)_{|C} < \tilde{\epsilon}_{|C}\}, \tag{6}$$

$$B_{max_C}(g, \epsilon; h, C) = \{g' : \max_C |g - g'|_h < \epsilon\}. \tag{7}$$

(Clearly it makes no difference to define the latter basis with the supremum instead of the maximum.) As before, $\tilde{\epsilon}$ is a positive continuous scalar field in the first and ϵ a positive number in the second, and their respective topologies $\mathcal{T}_C$, $\mathcal{T}_{max_C}$ are generated through arbitrary union over all choices of g_{ab}, h_{ab}, C and $\tilde{\epsilon}$ or ϵ.

Proposition 5. *For any M, $\mathcal{T}_C = \mathcal{T}_{max_C}$.*

Proof. Consider an arbitrary $B_C(g, \tilde{\epsilon}; h, C)$ and let $\epsilon = \min_C \tilde{\epsilon}$. Then clearly $B_{max_C}(g, \epsilon; h) \subseteq B_C(g, \tilde{\epsilon}; h, C)$ so by the lemma $\mathcal{T}_C \subseteq \mathcal{T}_{max_C}$. Conversely, consider an arbitrary $B_{max_C}(g, \epsilon; h)$ and define $\tilde{\epsilon} = \epsilon\tilde{1}$. Then clearly $B_C(g, \tilde{\epsilon}; h, C) \subseteq B_{max_C}(g, \epsilon; h)$, so by the lemma $\mathcal{T}_{max_C} \subseteq \mathcal{T}_C$. □

Not only do these bases define the same topology, but they also coincide with the open topology just when M is compact.

Proposition 6. *If M is non-compact, $\mathcal{T}_C \subset \mathcal{T}$.*

Proof. For any arbitrary $B_C(g, \tilde{\epsilon}; h, C)$, clearly $B(g, \tilde{\epsilon}; h) \subset B_C(g, \tilde{\epsilon}; h, C)$ so by the lemma $\mathcal{T}_C \subseteq \mathcal{T}$. Now, for the sake of deriving a contradiction, assume that for any $B(g, \tilde{\epsilon}; h)$ there is some $B_C(g, \tilde{\epsilon}'; h, C) \subseteq B(g, \tilde{\epsilon}; h)$. Consider

$$g'_{ab} = \left(1 + \sqrt{\frac{\rho}{h^{am}h^{bn}g_{ab}g_{mn}}} g_{ab}\right)$$

where ρ is any positive scalar field such that $\rho_{|C} < \tilde{\epsilon}_{|C}$ but $\sup_M \rho$ does not exist (i.e., is infinite). Clearly $g' \in B_C(g, \tilde{\epsilon}'; h, C)$ but $g' \ni B(g, \tilde{\epsilon}'; h)$, a contradiction. Thus by the lemma, $\mathcal{T} \nsubseteq \mathcal{T}_C$. □

Let $\mathcal{T}^k_{CO}$ be the topology generated from the subbasis of sets of the form

$$O^k(C, U) = g \in L(M) : j^k\hat{g}[C] \subseteq U, \tag{8}$$

where C ranges over all compact subsets of M and U ranges over all open sets of the manifold topology of $J^k(M, \hat{L}(M))$.

Proposition 7. *$\mathcal{T}^k_C(h) = \mathcal{T}^k_{CO}$ for any positive definite h.*

Proof. Similar to that of proposition 2, using the fact that one can use h to define a metric on the cross-sections $\hat{g}$. See (Munkres, 2000, Theorem 46.8, p. 285). □

Another idea for expressing the continuity of a parameterized family of spacetimes comes from Geroch (1969), who has proposed a way of interpreting certain limiting relations entirely geometrically through the continuity (smoothness, and so on) of certain fields. Roughly, in the simplest case of a one-parameter family, one constructs a five-dimensional manifold from the four-dimensional manifolds of the family stacked by their identifying parameter. One can generalize this notion to essentially arbitrary finite-dimensional parameterizations. (Cf. Fletcher (2015).) The notion of continuity this defines ends up being equivalent with the notion received from the C^k compact-open topologies, $\mathcal{T}^k_C$.

More precisely and generally, suppose that one is given a family of metrics $\{\overset{q}{g} \in L(M) : q \in N\}$, where N is a smooth, connected manifold.[7] Let $\mathcal{M}$ be a manifold such that there is some diffeomorphism $\phi : M \times N \to \mathcal{M}$, and let $\psi^{(q)} : M \to \mathcal{M}$ be a family of embeddings defined by $\psi^{(q)}(p) = \phi(p, q)$. Thus the field $\tilde{q} : \mathcal{M} \to \mathbb{R}$ that maps $\phi^{-1}(p, q) \mapsto q$ is smooth and labels the four-dimensional hypersurfaces foliating $\mathcal{M}$. One can then define on $\mathcal{M}$ a symmetric field Γ^{ab} with signature $(1, \dim M - 1, \dim N)$ by stipulating that $(\Gamma^{ab})_{|\phi^{-1}(p,q)} = (\phi^{(q)}_p)_*(\overset{q}{g}{}^{ab})$. In other words, Γ^{ab} is the field that on each $\tilde{q}$-constant hypersurface is just the pushforward of the inverse Lorentz metric $\overset{q}{g}{}^{ab}$. One can then find a derivative operator that is compatible with Γ^{ab} and $\nabla_a\tilde{q}$, i.e., $\nabla_a\Gamma^{bc} = \mathbf{0}$ and $\nabla_b\nabla_a\tilde{q} = \mathbf{0}$, and that makes these two fields orthogonal: $\Gamma^{ab}\nabla_a\tilde{q} = \mathbf{0}$. With this construction in place, we can say that the family $\{\overset{q}{g} \in L(M) : q \in N\}$ is *continuous in the C^k geometric sense* when the corresponding field on $\mathcal{M}$, Γ^{ab}, is everywhere continuous.

To show that the C^k geometric continuity of a family of metrics is equivalent to its continuity in the C^k compact-open topology, it will be helpful to use the following lemma (adapted from (Munkres, 2000, p. 287, Theorem 46.11)):

Lemma 2. *Let X and Y be topological spaces, and give the set of continuous functions from X to Y, denoted $\mathcal{C}(X, Y)$, the C^0 compact-open topology.*[8] *If $f : X \times Z \to Y$ is continuous, then so is the induced function $F : Z \to \mathcal{C}(X, Y)$ defined by the equation $(F(z))(x) = f(x, z)$. The converse holds if X is locally compact*[9] *and Hausdorff.*

The idea is to apply the lemma for $X = M$ and $Y = J^k(M, \hat{L}(M))$, i.e., when the set of continuous functions under consideration are the k-jets of Lorentz metrics. So, let $\mathcal{J}^k_C$ be the C^0 compact-open topology on this set. This topology is canonically bijective with the C^k compact-open topology $\mathcal{T}^k_C$.

Proposition 8. *$U \in \mathcal{J}^k_C$ iff $\pi^0_2[U] \in \mathcal{T}^k_C$.*

This follows as an immediate consequence of the relevant definitions. These two propositions can then be used to prove the aforementioned equivalence:

[7] Geroch does not require that the metrics be defined on diffeomorphic manifolds, but I can confine attention to that case here.

[8] I.e., that topology with the subbasis given by sets of the form $O(C, U) = \{f \in \mathcal{C}(X, Y) : f[C] \subset U\}$ for all compact $C \subseteq X$ and open $U \subseteq Y$.

[9] A topological space is locally compact when each point has a compact neighborhood.

Proposition 9. *A family of Lorentz metrics $\{\overset{x}{g}\}_{x\in X}$ on M parameterized by a smooth, connected manifold X is C^k continuous in the geometric sense iff it is continuous in the C^k compact-open topology.*

Proof. Let $n = dim(M)$ and $m = dim(X)$. Suppose that the family $\overset{x}{g}$ is C^k continuous in the geometric sense. Each $(n+m)$-dimensional inverse metric Γ^{ab} corresponds to a cross-section $\hat{\Gamma}$ of a bundle $\Gamma(M)$ of $(n+m)$-dimensional metrics over $\mathcal{M}$, whose partial derivatives to order k are encoded in the k-jet $j^k\hat{\Gamma}$. Then the smooth bundle map $\phi : J^k(M, \Gamma(M)) \to J^k(M, \hat{L}(M))$ induced by the projection $\pi : \mathcal{M} \to M$ can be composed with $\hat{Gamma}$ to yield the function $f = \phi \circ \hat{\Gamma} : \mathcal{M} \cong M \times X \to J^k(M, L(M))$, which is C^k because $\hat{\Gamma}$ is C^k by hypothesis. Lemma 2.2 then entails that the map $F : X! \to \mathcal{C}(M, J^k(M, \hat{L}(M)))$ defined by $F : x \mapsto j^k\hat{\overset{x}{g}}$ is continuous. But the range of F is just the set of k-jets of cross-sections of $\hat{L}(M)$ with the C^0 compact-open topology, which by proposition 8 is canonically bijective with the C^k compact-open topology on Lorentz metrics.

Conversely, suppose that the family $\overset{x}{g}$ is continuous in the C^k compact-open topology, or equivalently, that the map F defined above is continuous when $\mathcal{C}(M, J^k(M, \hat{L}(M)))$ is given the C^0 compact-open topology. Since M is locally compact and Hausdorff, lemma 2.2 entails that the map $f : M \times X \to J^k(M, \hat{L}(M))$ is continuous. Thus $\overset{x}{g}_{ab}(p)$ is jointly C^k in x and p.

Let $\psi^{(x)} : M \to \mathcal{M}$ denote the embeddings that yield the (inverse) metric Γ^{ab}, which is C^k when, for any smooth field α_{ab} on $\mathcal{M}$, $\alpha_{ab}\Gamma^{ab}$ is C^k. Now for any $p \in M$ and $x \in X$, $(\alpha_{ab}\Gamma^{ab})_{|\psi^{(x)}(p)} = (\psi_p^{(x)})^*(\alpha_{ab})\overset{x}{g}_{ab}$; by assumption $(\psi_p^{(x)})^*(\alpha_{ab})$ is smooth; and $\overset{x}{g}^{ab}$ is C^k because its inverse is. Thus Γ^{ab} is C^k. □

2.3 Point-Open Topologies

As we saw in the last section, the compact-open topology controls similarity between spacetimes by how those spacetimes differ on compact sets. An even weaker condition is to control similarity by how spacetimes differ at a finite number of points. The C^k *point-open* topologies $\mathcal{T}_P^k$ on $L(M)$ take their name from setting $\mathcal{S} = \{\{p\} : p \in M\}$, yielding the following subbasis (with the obvious but harmless abuse of notation):

$$O^k(U, p) = \{g \in L(M) : j^k\hat{g}(p) \subseteq U\}. \tag{9}$$

Following the other of the above outlined ideas, an alternative subbasis consists in the sets

$$B^k(g, \tilde{\epsilon}; h, p) = \{g' \in L(M) : (|g - g'|_h)_{|p} < (\tilde{\epsilon})_{|p}, \ldots, (|\nabla^{(k)}(g - g')|_h)_{|p} < (\tilde{\epsilon})_{|p}\}. \tag{10}$$

Both subbases essentially just pick out all the open sets within each fiber of the k-jet bundle $J^k(M, \hat{L})$, and because this is a finite-dimensional vector space, it carries a unique (Hausdorff) topology. Thus:

Proposition 10. *The sets $O^k(U, p)$ and $B^k(g, \epsilon; h, p)$ are subbases for the same topology on $L(M)$.*

It is clear that, in fact, a single choice of h will do, as different choices of h merely determine different norms in the fiber: again, each (true) norm generates the same topology in the fiber. Similar considerations reveal that a constant $\tilde{\epsilon}$ will do.

The point-open topologies control similarity amongst spacetimes by how they differ at a finite number of points. Although these topologies have not been discussed so explicitly in the literature, they are implicitly invoked in the convergence criteria of Malament (1986) and others in discussions of the Newtonian limit of general relativity. Malament declares that $\lim_{n\to\infty} \overset{n}{g}_{ab} = g_{ab}$ just when for every tensor field α^{ab}, $\lim_{n\to\infty} \alpha^{ab}\overset{n}{g}_{ab} = \alpha^{ab} g_{ab}$ at every $p \in M$. Note that each such α determines a fiber norm $|\cdot|_\alpha$, the total collection of which is separating. Thus these also must generate the same topology as those above, again due to the uniqueness of the Hausdorff topology for topological vector spaces.

3 Comparing Topologies and Methodological Contextualism

The differences between the open, compact-open, and point-open topologies turn on the degree of control the notion of similarity they encodes enforces. Thus one might think that the seeming "global" control of the open topologies is stronger than the others, analogous to the notion of uniform convergence from standard analysis and thus, perhaps, preferred over the others. But the conditions of convergence/continuity it entails turn out to be somewhat stronger than what one might expect from uniform convergence/continuity.

Proposition 11. *Let $g, \{\overset{n}{g}\}_{n\in\mathbb{N}}$ be Lorentz metrics on a non-compact manifold M. Then $\lim_{n\to\inf} \overset{n}{g} = g$ in the open C^k topology on $L(M)$ iff there is a compact $C \subset M$ such that:*

1. *for sufficiently large n, $\overset{n}{g}_{|M-C} = g_{|M-C}$; and*
2. *$\lim_{n\to\infty} \overset{n}{g}_{|C} \to g_{|C}$ in the compact-open C^k topology on $L(C)$.*

Proof. See Golubitsky and Guillemin (1973, pp. 43–4). □

Proposition 12. *Let X be any path-connected topological space, and suppose that the smooth manifold M is non-compact. Then $f : X \to L(M)$ is continuous when $L(M)$ is given the C^k open topology iff $F(x_1, x_2) = f(x_1) - f(x_2)$ has compact support (for each fixed x_1, x_2) and is continuous when $T^0_2(M)$ is given the C^k compact-open topology.*

(One direction of the proof follows from (Fletcher, 2015, Proposition 2), and the other is a straightforward calculation.)

Not only do these propositions exhibit a connection between the open and compact-open topologies, they also show that the open topologies are much finer than it may initially appear, making it rather difficult for sequences of spacetimes to converge and parameterized families of spacetimes to vary continuously. Indeed, this difficulty exhibits a connection with another topology, more rarely used, called the (C^0) fine topology (Hawking, 1971), $\mathcal{T}_F$. It can be given a subbasis of the form

$$B_F(g, \tilde{\epsilon}; h, C) = \{g' \in L(M) : g'_{|C} = g_{|C}, |g - g'|^2_h < \tilde{\epsilon}\} \tag{11}$$

Each neighborhood of g consists of metrics that differ from g only on a compact set. We thus have an immediate corollary to the above propositions:

Corollary 2. 1. *Let $g, \{\overset{n}{g}\}$ be metrics on a non-compact manifold. Then $\lim_{n\to\infty} \overset{n}{g} = g$ in the (C^0) open topology iff it does does in the (C^0) fine topology.*

2. *Let* $\{\overset{x}{g}\}_{x\in X}$ *be a family of metrics on a non-compact manifold parameterized by a path-connected manifold* X*. Then this family varies continuously in the* (C^0) *open topology iff it does so in the* (C^0) *fine topology.*

However, the fine topology is in fact strictly finer than the open topology:

Proposition 13. *If* M *is non-compact,* $\mathcal{T}(h) \subset \mathcal{T}_F$.

Proof. First note that the sets $B'_F(g, \tilde{\epsilon}; h) = \bigcup_C B_F(g, \tilde{\epsilon}; h, C)$ form a neighborhood basis of g for the fine topology. So consider an arbitrary $B(g, \tilde{\epsilon}; h)$ and note that if $g' \in B_F(g, \tilde{\epsilon}; h, C)$, then $g' \in B'_F(g, \tilde{\epsilon}; h)$. Hence $B'_F(g, \epsilon; h) \subseteq B_F(g, \epsilon; h)$ and by lemma 2.2, $\mathcal{T}(h) \subseteq \mathcal{T}_F$.

Conversely, suppose for the sake of contradiction that for every $B'_F(g, \tilde{\epsilon}; h)$, there is some $B(g, \tilde{\epsilon}; h) \subseteq B'_F(g, \tilde{\epsilon}; h)$. Consider

$$g'_{ab} = \left(1 + \sqrt{\frac{\tilde{\epsilon}'/2}{h^{am}h^{bn}g_{ab}g_{mn}}}\right) g_{ab},$$

noting that $g' \in B(g, \tilde{\epsilon}'; h)$ but $g' \neq g$ everywhere. Thus $g' \notin B_F(g, \tilde{\epsilon}; h, C)$ for each C, hence $g' \notin B'_F(g, \tilde{\epsilon}; h)$. Therefore by lemma 2.2, $\mathcal{T}_F \not\subseteq \mathcal{T}(h)$. □

This may at first be counterintuitive. How can it be that distinct topologies agree on which sequences converge? It turns out the difference turns on a certain subtlety of topological spaces and their relations to convergence classes. To show how this is the case, we need some more terminology.

A topological space X is said to have a *countable basis at* $x \in X$ if there is a countable collection $\{U_i\}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the U_i. If X has a countable basis at each point, it is said to be *first countable.* X is said to be *metrizable* if there is a continuous function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) = 0$ iff $x = y$;

2. $d(x, y) = d(y, x)$; and

3. $d(x, z) \leq d(x, y) + d(y, z)$.

When a topological space X is metrizable, it is first countable: just take the countable basis at each point $x \in X$ to be the ϵ-balls $\{y \in X : d(x, y) < \epsilon_n\}$ with $\epsilon_n = 1/n$. If a space is first countable, then its topology is entirely determined by its convergent sequences, but the converse need not be true (Kelley, 1955, Theorems 8–9, pp. 72–4).[10]

Proposition 14. $\mathcal{T}(h)$ *is first-countable iff* M *is compact.*

Proof. See (Golubitsky and Guillemin, 1973, pp. 43–44). □

Corollary 3. $\mathcal{T}$ *is metrizable iff* M *is compact.*

[10] Generally, one can characterize any topology in terms of convergence of nets, i.e., directed sequences of sets. (A sequence is a special case of a net in which all of the sets are singletons.) But in this paper we shall have no need to do so.

Proof. That M being non-compact implies that $\mathcal{T}$ is not metrizable is immediate, so consider the case where M is compact. Then one can check that the standard uniform metric $d_h(g, g')$ suffices. □

One may interpret the failure to be first-countable as there being more than countably many ways for two metrics to differ on a non-compact manifold. Accordingly, there are "too many" ways for a pair of Lorentz metrics to differ than a distance function (i.e., a function satisfying the triangle inequality) could allow.

On account of these features, one might only require "control" over compact subsets of spacetimes, but the compact-open topologies have their own peculiarities.

Proposition 15. *If* $\dim(M) \geq 3$, *then chronology violating spacetimes are generic in* $L(M)$ *in any of the* C^k *compact-open topologies.*

Proposition 16. *If* $\dim(M) \geq 3$, *no Lorentz metric is stably causal in any of the* C^k *compact-open topologies on* $L(M)$.

Proposition 17. *Every space-time* (M, g) *containing a closed timelike curve does so stably in any of the* C^k *compact-open topologies if* $\dim(M) \geq 3$,.

(Proofs of these propositions are exactly analogous to those in Fletcher (2015).) Thus, when it comes to situations where global control is really needed, such as determining the stability of a global (= not local) property of spacetime, the compact-open topologies are not of use. Switching to any of the point-open topologies does not change this, and further, these topologies in general render strictly *more* than the geometrically continuous families of spacetimes continuous.

These antimonies dissolve, however, if one does not require a single topology to be associated to the models of general relativity. Instead, one can be a methodological contextualist (Fletcher, 2015): depending on the nature and scope of a particular research question asked, one can justify a choice of topology relevant for the details of that question. For example, questions concerning global properties might use (something like) an open topology, while questions concerning properties confined to bounded regions or points might use the compact-open or point-open topologies, respectively. If one reminds oneself that the choice of topology just formalizes a notion of similarity among spacetimes, this is quite natural: like any other class of sufficiently complicated structures, spacetimes can be similar in various, sometimes incompatible ways, so one must specify with enough precision the notion of similarity that might be relevant for a particular research question.

References

Fletcher, Samuel C. "Similarity, Topology, and Physical Significance in Relativity Theory," *British Journal for the Philosophy of Science* (forthcoming, 2015).

Geroch, R. "Limits of Spacetimes," *Communications in Mathematical Physics* 13: 180–93 (1969).

Geroch, R. "Singularities," *Relativity.* Ed. Moshe Carmeli, Stuart I. Fickler, and Louis Witten. New York: Plenum, 1970.

Geroch, Robert. "General Relativity in the Large," *General Relativity and Gravitation* 2.1: 61–74 (1971).

Golubitsky, M. and Guillemin, V. *Stable Mappings and their Singularities*, New York: Springer, 1973.

Hawking, S.W. "The Existence of Cosmic Time Functions," *Proc. R. Soc. Lon. A* 308: 433–435 (1969).

Hawking, S.W. "Stable and Generic Properties in General Relativity," *General Relativity and Gravitation* 1.4: 393–400 (1971).

Hawking, S.W. and G.F.R. Ellis. *The large scale structure of space-time.* Cambridge: Cambridge University Press, 1973.

Kelley, John. *General Topology.* New York: Van Nostrand, 1955.

Lerner, David E. "The Space of Lorentz Metrics," *Communications in Mathematical Physics* 32, 19–38 (1973).

Malament, David B. "Newtonian Gravity, Limits, and the Geometry of Space," *From Quarks to Quasars*. Ed. Robert G. Colodny. Pittsburgh: University of Pittsburgh Press, 1986.

Munkres, James R. *Topology.* 2nd ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2000.

Schaefer, H.H. *Topological Vector Spaces*. 2nd ed. With M.P. Wolff. New York: Springer, 1999.