Space invaders and (small) exotic sources

Abstract

We discuss similarities between the space invaders scenario in Newtonian mechanics and spacetimes obtained using small exotic smooth structures on the topological manifold $\mathbb{R}^4$ in classical general relativity.

1 Space invaders

The space invaders scenario in Newtonian mechanics describes a situation in which determinism — understood as a thesis that fixing physical situation of the system in one moment of time amounts to fixing it at all times — is violated. In this scenario one usually starts with the system of finitely many bodies which due to interactions between them (the easiest construction uses collisions) accelerate without bound, "escaping" to infinity in finite time. Since Newtonian mechanics is time-reversal invariant, the system's reverse temporal evolution also is a Newtonian system, in which bodies appear "out of nowhere" at an arbitrary moment of time.

Since the scenario makes use of unbounded acceleration, it is relativistically untenable. One may thus wonder whether any similar scenarios are allowed by classical general relativity. Indeed, situations somehow resembling space invaders can happen in spacetimes with timelike infinity, such as anti-de Sitter spacetime. But this, in turn, can be remedied by postulating the boundary conditions. Are there situations similar to space invaders which cannot be remedied in this way? We suggest that if one exploits exotic smooth manifolds the answers is positive.

2 What are exotic $\mathbb{R}^4$?

A smooth manifold M' is called *exotic* if it is homeomorphic but not diffeomorphic to a given manifold M (conventionally, "usual" smooth manifolds, such as n-spheres and $\mathbb{R}^n$ are the non-exotic ones). It is far from obvious that any exotic manifolds exists. But in 1956 Milnor constructed first exotic manifold, the exotic

7-sphere, and a number of mathematical results followed, including classification of exotic spheres in higher dimensions.

In case of $\mathbb{R}^n$, for $n \neq 4$ there exists no exotic smooth structure. But in case of $n = 4$ uncountably many exotic smooth structures exist. We will denote them by $\mathbb{R}^4_\Theta$ and the non-exotic smooth structure as $\mathbb{R}^4$. $\mathbb{R}^4_\Theta$ divide into two classes: small, which can be smoothly embedded into $\mathbb{R}^4$, and large, which cannot.[1].

Starting with a series of papers by Brans in early 1990s, exotic $\mathbb{R}^4$'s have been used (among others) in quantum field theory, inflationary cosmological models, and quantum gravity. See (Asselmeyer-Maluga and Brans, 2007) for an overview of exotic manifolds directed at physics audience and (Asselmeyer-Maluga and Brans, 2015) for some recent developments. In contrast, we are interested in the classical general relativity, in particular, whether any philosophically interesting morals can be drawn from the existence of exotic smooth structures.

3 "Exotic invaders"

Here we show how to use $\mathbb{R}^4_\Theta$ to construct scenario we dub "exotic invaders". Then we discuss its physical and philosophical relevance. We assume that the relativistic spacetime is a pair $\langle M, g\rangle$, where M is a four dimensional second countable smooth manifold, and g is Lorentz-signature pseudo-metric.

3.1 Construction and similarity to space invaders

In $\langle \mathbb{R}^4, g\rangle$ fix Cauchy surface Σ. Find open subset C above Σ. Remove C and insert small exotic $\mathbb{R}^4_\Theta$. Define g' in any way on $\mathbb{R}^4_\Theta$. "Folklore" knowledge is that any non-compact smooth manifold admits a Lorentzian metric, so such g' always exists.[2]

Then the result of (Kokkendorff, 2002), that for any non-compact smooth manifold there exists a Lorentzian metric with a smooth time function, can be used to establish stably causal metric on a small $\mathbb{R}^4_\Theta$.

However, $\mathbb{R}^4_\Theta$ are not globally hyperbolic. Poincare conjecture — now known to be true — implies that $\mathbb{R}^4_\Theta$ is not a product of $\mathbb{R} \times_{smooth} \mathbb{R}^3$; but by a theorem due to Dieckmann (Dieckmann, 1988), every globally hyperbolic spacetime can be represented as $\mathbb{R} \times_{smooth} \Sigma$, for Σ a Cauchy surface.

[1] These constructions can be found in (Scorpan, 2005)

[2] What we mean by "folklore" here is that the author is not aware of a better proof of existence of Lorentzian metric on non-compact manifolds than the one presented by Geroch and Horowitz (Geroch and Horowitz, 1979). Their proof, however, predates discovery of $\mathbb{R}^4_\Theta$; and thus the author is not certain whether some subtelties might not block Geroch and Horowitz argument after all, due the failure of smooth connected-sum-splitting (responsible for some of $\mathbb{R}^4_\Theta$).

Moreover, since there is no diffeomorphism d between $\mathbb{R}^4$ and $\mathbb{R}^4_\Theta$, $g' \neq d_*(g)$, so $\langle \mathbb{R}^4, g \rangle$ and thus $\langle \mathbb{R}^4_\Theta, g' \rangle$ are not isometric.

To sum up, we constructed two spacetimes: $\langle \mathbb{R}^4, g \rangle$ and $\langle \mathbb{R}^4_\Theta, g' \rangle$ which are isometric up to a given Cauchy surface Σ, but are no isometric above, with $\langle \mathbb{R}^4, g \rangle$ continuing in a globally hyperbolic way, and $\langle \mathbb{R}^4_\Theta, g' \rangle$ containing structure which exists due to small exotic smoothness, is localized, and is non-isometric to the globally hyperbolic developement. Due to freedom of choice of the small $\mathbb{R}^4_\Theta$ (since there are uncountable many possible choices of the smooth region Θ), we obtain a large class of spacetimes with "exotic invaders".

The parallel with the space invaders is that the "exotic invader" appears at an arbitrary moment, without data on Σ determining whether the invader will appear in the spacetime or not. This situation, however, differs from the space invaders scenario in important two ways. First, no recourse to time-reversal invariance is needed. Second, the "exotic invader" does not come from any "infinity", and *prima facie* seems to not involve any acceleration at all.

Note also that since $\mathbb{R}^4_\Theta$ is homeomorphic to $\mathbb{R}$, we obtain indeterministic-looking scenario which is hole free in the intuitive sense that no points are removed from the spacetime manifold M.

3.2 Brans conjecture

We have used $\mathbb{R}^4_\Theta$ to construct a new spacetime $\langle \mathbb{R}^4_\Theta, g' \rangle$. But one may object that $\langle \mathbb{R}^4_\Theta, g' \rangle$ is merely a mathematical construct, and demand an interpretation in physical terms. Can one provide such an interpretation for the exotic smooth structure?

It turns out that (Brans and Randall, 1993) suggested such an interpretation:

> $\mathbb{R}^4_\Theta$ acts as a gravitational source.

This could be understood as:

> Any Lorentzian metric on $\mathbb{R}^4_\Theta$ is non-flat in the exotically smooth regions.

Thus, it seems that if Brans conjecture is true in classical general relativity, there are spacetimes with spontaneously appearing gravitational sources due to exotic smooth structure of spacetime. This provides a minimal interpretation of $\langle \mathbb{R}^4_\Theta, g' \rangle$.

4 Summary

Using small exotic $\mathbb{R}^4_\Theta$ we constructed class of spacetimes with behaviour similar to Newtonian space invaders. We end with a remark on cosmic censorship.

Earman (Earman, 1995) noted that cosmic censorship conjecture rules out spacetimes with exotic smooth structure. Inspired by the progress in the formulation of the (notoriously wooly) conjecture (see (Ringström, 2009)), and assuming that Brans conjecture holds, we suggest the following particular case of the cosmic censorship might be worth investigating:

(*) any Lorentzian metric which is non-flat due to exotic smoothness leads to Catastrophe,

(with the Catastrophe being either physically unrealistic in the sense of violating some interpretative assumptions, such as energy conditions or some structural properties, or instability under perturbations). It is an open question whether for various possible meanings of the Catastrophe (*) could be established. Note also that our reading of Brans conjecture might allow for the weak version of the censorship to hold, shielding off the exotic regions by black holes.

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