Modality, Mathematics, and Time; A common flaw in modal arguments.

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Abstract

Modality in the context of mathematical discourse has received increased attention recently. In this paper I analyse a kind of modal argument that can be brought against the position that there is a single, determinate, maximal interpretation of set-theoretic discourse. I argue that attempts to deepen Naturalistic problems through modal considerations are closely analogous to Gödel's argument from General Relativity Theory against the reality of Time. Both I argue fall flat against their targets for similar reasons; an imprecision in the modality used in the arguments results in principles that appear true, but are unlikely to be jointly accepted with the *same* modality in play by the parties in question. I conclude with some observations concerning the kinds of possibility at issue, and argue that the problems posed by forcing extensions and proper classes are not deepened by modal considerations.

Introduction.

Recent debates in the Foundations of Mathematics have often focussed on the rôle of *mathematical practice* and how this might be mobilised in favour of certain views. In this paper, I analyse some arguments concerning the nature of mathematical practice, and some *modal* principles that one might argue deepen these problems. We shall see that there is a substantial and instructive analogy between the kinds of arguments presented in the Philosophy of Set Theory, and Gödel's argument against the reality of Time.

My strategy is as follows. §1 provides a characterisation of the positions at issue in the Philosophy of Set Theory, and reviews some salient features of the view (Universism) under attack. §2 then outlines the objections against the Universist from proper classes and forcing extensions. §3 notes that the argument requires some bolstering to be effective, and considers one line of argument from *modality*. §4 notes that the argument presented bears a very close resemblance to one made from relativity theory against the existence of time, and explains one way of resisting the argument. §5 then shows that parallel moves may be made in the mathematical case. It is concluded that the argument of §3 does not represent a significant advancement in the dialectic and does not further threaten Universism.

1 Universism, Multiversism, and Mathematical Practice.

We first begin with some key notions, useful for setting up the dialectic. Given that the debate at hand attempts to mobilise mathematical practice in drawing philosophical conclusions, we make the following assumption:

> [Moderate Naturalism] If we have two philosophical positions X and Y, and a piece of mathematical discourse Φ, if X can give a better interpretation of Φ than Y, then X is, in this respect, a better philosophy than Y.

Of course, what constitutes a 'good' or 'better' interpretation is a significant philosophical issue. There are two *desiderata* relevant for the current discussion:

1. *Preservation of Truth Value*. It is preferable to be able to interpret set theorists in such a way as to make their statements *true* (under our interpretation).
2. *Intent.* Interpretations that respect the *intended* in reference of the set-theoretic speaker are better than those that have to drastically reinterpret the objects about which the speaker intends to speak.
3. *Character.* We wish to provide an interpretation that respects the *phenomenological character* of mathematical reasoning.

A full defense of these constraints is outside the scope of the current paper. However, we can see that each is individually *prima facie* plausible (given that we are already in the business of interpreting set-theoretic practice as faithfully as possible). Moreover, they serve to provide a backdrop against which we may analyse different views and be precise about their respective strengths and weaknesses. For the purposes of the debate at hand then, they are useful for seeing the exact sense in which we might assert that one view is better than another.

The view under attack from mathematical practice can be characterised thus:

> [Universism] There is a unique, maximal, interpretation of set-theoretic discourse (denoted by 'V') under which every sentence of Set Theory receives a determinate truth value.

This contrasts strongly with the following:

> [Multiversism] There is no maximal, unique interpretation of set-theoretic discourse. Rather, there are many, equally legitimate such interpretations.

The term 'Multiversism' refers to a variety of positions, dependent on which interpretations are to count as legitimate. We might, for instance, take any model of first-order ZFC to be legitimate, thereby countenancing the existence of any model-theoretic construction from a particular starting universe (*exempli gratia* forcing extensions, ultrapower constructions, non-well-founded models *et cetera*). Alternatively, we might countenance all models of ZFC_2, resulting in universes of the form V_κ for strongly inaccessible κ.

We should first note that Universism appears to be, and has been widely regarded[1] as, the more pre-theoretically appealing view. In order to clarify the dialectic, we briefly canvass a couple of explanations for this fact.

1. The Iterative Conception of Set. Most are familiar with the Iterative Conception of Set, under which we form sets (*via* the power set operation) in stages through the ordinals. Given then the pre-theoretically appealing thought that we know what the

[1] See, for example, [Linnebo, 2010], [Shapiro and Wright, 2006] among many others.

ordinals are and what the power set of a particular set is, we arrive at a determinate structure by iterating the power set operation through all the ordinals. Post-theoretically, such an argument turns out to be extremely controversial; it being up for grabs whether either talk of *all* subsets or *all* ordinals is cogent. However, the pre-theoretic appeal lives on (see, for example, [Koellner,]). Multiversist views, on the other hand, have to then contend with the fact that the *Intent* of Universist-minded set theorists is to refer to such a determinate structure.

2. Making sense of quantificational practice. The Universist is easily and transparently able to account for the quantificational practice of both laypersons and set-theorists. Simply put, when a speaker uses the term 'all sets', the Universist is able to interpret her with the minimum of fuss as concerned with V. Multiversist views, on the other hand, have to provide an account of such talk given that there is no maximal interpretation of set-theoretic discourse, possibly incurring Intent and even Preservation of Truth value based costs. Such issues are well-worn in the literature on Absolute Generality, and we do not revisit them here. It suffices to note that, *prima facie*, this is a virtue of adopting Universism.

None of the above reasons should be taken as definitive arguments in favour of Universism or against Multiversism, and in fact there is a burgeoning literature and cogent responses in each case. Nor is the above enumeration of pleasing theoretical features of Universism exhaustive. All the reader need take from the above considerations for the rest of the paper is that Universists have pre-theoretic and intuitive reasons for adopting their position, and Multiversists have some challenges to face. The fact that some have found these kinds of reasons to be compelling is witnessed by the fact that some philosophically engaged contemporary set theorists are Universists (such as Woodin, Welch, and Koellner, among others). In the context of the dialectic, and given subsequent arguments, the fact that Universists take their position to be *prima facie* motivated will turn out to be significant.

2 The Arguments Against the Universist.

Despite these pleasing features of Universism there are many arguments that have been pressed against the view on the basis of Naturalism. We examine to just two here, however they are two that are very closely aligned with *mathematical practice*. The Universist has her pre-theoretic reasons for her position. The challenge for the Multiversist is to dislodge her confidence in her view by forcing her to admit significant Naturalistic cost, and argue that her position does not suffer from these problems. We shall see that in the end that these mathematically sophisticated arguments, when given a modal spin, do not represent a substantial philosophical advance in arguing against the Universist position.

2.1 Proper Classes.

The first concerns a well-known problem for the Universist; namely the issue of proper classes. Since the discovery of the set-theoretic paradoxes at the turn of the 20th Century, philosophers and mathematicians have been aware that there appear to be collections on the Universist picture (such as 'the collection' of all Ordinals or 'the collection' of all non-self-membered sets), that (if they were sets) would produce contradictions. Again there is an expansive extant literature on interpreting proper class discourse[2] for the Universist, and one assumes that if she has made it this far then she has made her peace with these philosophical problems. Rather than revisiting this well-established literature, we will examine whether or not a Universist should be threatened by proper classes from a *Naturalist* perspective.

Whence the problem? Simply put proper classes, and indeed *relationships between* proper classes are often the subject of discussion amongst set theorists. Consider the following popular definition of a *measurable cardinal*:

Definition 1. A cardinal κ is *measurable* iff it κ the critical point of a non-trivial, elementary embedding $j : V \longrightarrow \mathfrak{M}$ for some inner model $\mathfrak{M}$.

We remark very briefly on the definition in order to see where the problem lies. Here V is the universe of all sets and $\mathfrak{M}$ is a transitive inner model (*id est* $\mathfrak{M} \neq V$ and $\mathfrak{M}$ contains all ordinals). j is a mapping from V into $\mathfrak{M}$, to say that j is non-trivial is just to say that it is not the identity map, and to say that it is elementary means that it preserves first-order truth. Whence the problem? Well given that j is effectively a kind of function between two proper classes, it looks like a second-order object (*id est*, if we let 'Ω' denote the order type of the ordinals, j would be formed at $V_{\Omega+1}$). But this is problematic; while the Universist may have made her peace with proper class discourse, she *certainly* cannot countenance ordinals 'beyond' all the ordinals! How is she to interpret this discourse?

The standard method is to note that j admits of characterisations that do *not* require the examination of $V_{\Omega+1}$. We may provide equivalent first-order formulations of j by coding its behaviour using a syntactic formula ϕ, or alternatively talking about completeness properties on ultrafilters (a particular kind of set)[3]. However, the question remains that we apparently simulations (amenable to the Universist) that provide model-theoretic content (in the sense that we can render claims in such a way that they are satisfied) to objects which cannot be in V. Shapiro and Wright, in the following purple passage, jump on the issue enthusiastically:

> "Typically, the way around the 'annoying' meta-mathematical problems to which Schimmerling [in a survey article on a field[4] related to elementary embeddings] refers is to replace the long transfinite recursions with codings. That is, the set theorist works hard to simulate the long transfinite recursion within ordinary, first-order set theory. Nevertheless, it seems to us that this grand transfinite recursion is coherent as it stands, or at least as coherent as any thing else in set theory." ([Shapiro and Wright, 2006], p290)

So we see the problem, recursions that define objects 'past' the 'end' of V, seem to have a cogent theory as given by their codings under on the Universists view. Thus, the Universist immediately runs into problems with Intent and Character from a Naturalistic perspective, we have to reinterpret the intended talk of set theorists; the objects in question are neither what some set theorists Intend to refer to nor does reasoning involving the objects (such as Ultrafilters) have exactly the same Character as the reasoning concerning embeddings (despite the class-sized codings providing a *very* close interpretation). There is much more to be said about how exactly to

[2] Indeed, options for interpreting proper class talk on a Universist picture are available in [Boolos, 1984] (with subsequent development in [Uzquiano, 2003]), and [Horsten and Welch,].

[3] The metamathematical issues here are actually rather subtle. The above exposition is how the issue has largely been characterised in the literature (see [Shapiro and Wright, 2006]) and is satisfactory for seeing the rough outline of the problem.

[4] The field in question being the elegant yet extremely technically challenging field of Fine Structure Theory and Mice.

provide further content to the problem. First, we consider a different problem that has been posed for the Universist.

2.2 Forcing Extensions.

Similar issues apply for considerations of 'width'. Forcing is a construction by which we 'add' sets not already in a model $\mathfrak{M}$ to $\mathfrak{M}$. Again we suppress technicalities for clarity, but the important fact to note is that using a partial order $\mathbb{P} \in V$, we can define an object $G \subseteq \mathbb{P}$ such that $G \notin \mathfrak{M}$ (for non-trivial forcing constructions), then add G to $\mathfrak{M}$, close under the operations definable in $\mathfrak{M}$, and thereby form the forcing extension $\mathfrak{M}[G]$. For the sake of clarity, we repeat the *most* philosophically important feature of forcing for the debate at hand; $G \notin \mathfrak{M}$ for non-trivial forcing constructions.

The Naturalistic problem is generated by observing that often set theorists will use the term 'V' to denote the ground model over which we force. If this is actually taken to denote V proper, we have a *prima facie* problem; the generic G cannot be in V, but V is meant to be *all* the sets there are. The problem is mitigated somewhat by the fact that often the use of the term 'V' is patently an abuse of notation, implicitly referring either to a countable transitive model or a Boolean-valued description of V[5]. However, more recently Hamkins has provided what he calls the *Naturalist Account of Forcing*:

> "**Theorem 3.1** (Naturalist Account of Forcing). If V is a (the) universe of set theory and $\mathbb{P}$ is a notion of forcing, then there is in V a [first-order definable from a parameter] class model of the theory expressing what it means to be a forcing extension of V." ([Hamkins, 2012], p423)

The construction involved is somewhat technical[6] however the point is clear. We can, in definable first-order terms, express what it is to be a forcing extension of V within V. Indeed we can interpret much of the theory of $V[G]$ within this first-order definable model.

Thus we see a similar problem as in the proper classes case. We can provide a model-theoretic interpretation of a good deal of theory of $V[G]$ within V, in a manner that *very closely* resembles $V[G]$. Hence, we seem to have a cogent theory for talking about objects outside V. This puts Naturalistic pressure on the Universist, again *via* the Intent constraint. Moreover the models in question *look exceedingly* like V.

3 The Modal Argument.

We should pause here for some reflection, however. The above arguments proceed in a two step process:

1. Show that there are, for the Universist, claims about objects within V that model-theoretically simulate claims about objects that would have to be outside V were they to exist.

[5] See [Koellner,] for an exposition of these issues, and [Jech, 2002] for a description of the relevant technical facts.

[6] An exceptionally rough outline of the details (available in [Hamkins and Seabold, 2012]): Take an Ultrafilter U on the forcing algebra $\mathbb{B}$ and use it to map V into an elementary extension given by an inner model (denoted by $\check{V}_U$). Then, using U, form the quotient structure $V^{\mathbb{B}}/U$ of the Boolean-valued model $V^{\mathbb{B}}$. It turns out that, when examining the image of $\mathbb{B}$ in $\check{V}_U$, that $V^{\mathbb{B}}/U$ is exactly the forcing extension of $\check{V}_U$ by U.

2. Argue that this puts Naturalistic pressure on the Universist.

(1.) I take it is not up for dispute; it is a simple mathematical fact that we can think of the objects either in the Universe-acceptable manner or the 'extra-V' manner, and the security of the relevant theorems is not thereby threatened. What the Universist is likely to assert is that the 'extra-V' manner of thinking operates as a mere heuristic for getting at truths concerning objects within V.

The only place for the Universist to resist then is against (2.). Indeed, there is a peculiar feature of the two above arguments, both Hamkins and Shapiro and Wright motivate the issue by arguing that the relevant codings (acceptable to the Universist) of the 'extra-V' claims get *very close* to capturing the content of the Set Theory (particularly salient are codings of embeddings using first-order definable ϕ and the Naturalist Account of Forcing). However, while such codings serve to make the problem more vivid to the Universist, they also serve to decrease the Character-based cost she has to suffer. Given then, that there is *some* Naturalistic price to be paid by both Multiversist and Universist (largely Intent based), we wish to ask what makes the Universist position so bad? An additional argument is desirable.

We might begin to flesh out an argument as follows. Given that the Universist can simply put their foot down and regard the kinds of thinking given above as mere heuristic, we need additional philosophical arguments to show that these model-theoretic simulations, *via* plausible philosophical claims, motivate deep problems for her position. One way to do so is to use *modal* principles about mathematical reality.

The following is a principle to which the Universe Theorist is *very* likely to adhere:

> [Necessity] Mathematical objects exist out of necessity if at all.

Such a view is held by most, if not all, Universists[7]. We now simply need a principle that links the model-theoretic simulations and possibility.

> [Simulation] If a view A accepts some entities xx, and it is possible to provide a model-theoretic interpretation of a theory T using the xx (in the sense of being able to satisfy the claims of T with an interpretation that makes use of the xx) then A should accept the possibility of the objects that we attempt to refer to with T.

We are now in a position to run the following argument:

> (P1) Objects outside of V can be given model-theoretic interpretation within V.
>
> (P2) By [Simulation] the Universist should accept the possibility of such objects with their intended interpretation.
>
> (P3) By [Necessity] the Universist should accept the actual existence of objects outside V.
>
> (C) The Universist should accept the existence of objects outside V, and hence the should accept the falsity of her position.

Should the Universist accept [Simulation]? Considerations from the History of Mathematics indicate that she should accept *something like* [Simulation]. For example, Hamkins makes the following cogent point:

[7] With the possible exception of impure sets (as in [Fine, 1981]), but we set aside this issue here.

> "Historically, $\sqrt{-1}$ was viewed with suspicion, and existence deemed imaginary, but useful... Eventually, of course, mathematicians realized how to simulate the complex numbers $a + bi \in \mathbb{C}$ concretely inside the real numbers, representing them as pairs (a, b) with a peculiar multiplication $(a, b) \circ (c, d) = (ac - bd, ad + bc)$.= This way, one gains some access to the complex numbers, or a simulation of them, from a world having only real numbers, and full acceptance of complex numbers was on its way." ([Hamkins, 2012], p420)

Indeed the point is well-taken, such an argument indicates the way to understand the Complex numbers as a Field given by a number plane rather than number line. If the Universist is to deny [Simulation], then she must provide cogent reasons as to why [Simulation] is illegitimate in the problematic cases, but legitimate when trying to determine the existence of less esoteric mathematical entities. [Necessity] (as stated) is accepted by almost all Universists. The problem is thus well-posed.

Despite this, I think the argument fails against the Universist. The reason concerns the relevant interpretation of the modality. A more fine-grained conception of the modality, I shall argue, enables the Universist to provide a more subtle response to the problem, on which *neither* [Simulation] *nor* [Necessity] is clearly true. It will turn out that if we are to interpret the modality in such a way that [Necessity] comes out as true, [Simulation] is already obviously false for the Universist. I will conclude that insofar as the Universist already had an acceptable position *prior* to the modal argument, they continue to have a satisfactory position. But first, we should take our time to consider an analogy with a different area of Philosophy, and what can be learned there.

4 Gödel, Einstein, and Time.

We have to go to the Philosophy of Time to find an interesting analogy with the present case. The thesis that time exists shares several similarities with Universism. We may characterise the view as follows:

> [The Reality of Tensed Time (henceforth '$[RTT]$')] Tensed Time is real, in the sense that, given any two speakers x and y, their claims about the past, present, and future are either true or false.

$[RTT]$ seems initially obvious, but as is well-known has received a good deal of criticism. Indeed, it is far stronger than it initially appears. The view requires, that for any two or more speakers (who, let us assume, can travel exceptionally fast despite their fleshy composition and thus occupy radically different reference frames), given some tensed, otherwise bivalent[8], claims made by those speakers, there is a fact of the matter about whether the speakers are true or false. In order to do this, there needs to be a successive series of absolute frames of reference from which such claims are adjudicated.

The view is somewhat analogous to Universism in that it postulates the existence of single, absolute, maximal reference frames against which all other reference frames are to defer (given a speaker within a reference frame). Similarly, the Universist postulates a single, maximal, interpretation of set-theoretic discourse. Moreover,

[8] We make the stipulation that the claims to be tensed have to be bivalent in order to avoid a violation of $[RTT]$ for reasons unrelated to tense. For example, given a particular account of vagueness "Borderline Barry was bald" might fail to be bivalent in virtue of his borderline-baldness, rather than any tense-related issues.

as agents in the world we have no way of identifying the absolute interpretation in either case. The Universist, even if we grant[9] *full* second-order resources cannot fix the reference of V up to isomorphism[10]. Similarly, the $[RTT]$-theorist, given that she is always bound to a particular frame of reference, has no way of empirically observing whether two events are absolutely simultaneous[11] relative to a reference frame, and therefore has no empirical means for discovering the absolute reference frame.

The adherents of both views then use different *philosophical* reasons for motivating their position. Some of the Universists motivations have already been discussed. $[RTT]$-theorists often appeal to different reasons, for example that we know what it is to be present, or that tensed claims are needed for the successful navigation of the world[12].

Even more interestingly, modal objections have also been raised against the $[RTT]$-theorist. We shall see that extant arguments against this objection, and an analogy between the two situations, points a way out for the Universe Theorist, at least insofar as she had a satisfactory theory prior to the modal objection.

4.1 Gödel's Modal Objection.

A few expository details are necessary first. Given acceptance of our current best physical theory, we note that the widely accepted current best theory of space-time is given by the *General Theory of Relativity* (GTR), and associated Einstein Field Equations (EFEs) that hold of the spacetime structure. It is a difficult question (with a vast and highly technical literature[13]) whether there is a non-arbitrary way of defining successive absolute frames of reference. Given the current dialectic, this is a somewhat moot point; the $[RTT]$-theorist feels that she already has good philosophical reasons to accept the existence of such frames.

However, Gödel called into question, *via* a modal argument using $[RTT]$, the reality of time. Gödel's argument roughly proceeds as follows. We begin by considering the *Gödel Metric*; a model of the EFEs under which, for any observer x within the model there is a closed timelike curve through x[14]. Essentially in the model, it is possible (by travelling *very* fast, for a *very* long period of time), for an observer to travel into her future and end up in her past. Under such a model, the definition of an absolute reference frame for tensed statements is clearly impossible; there are

[9] It bears mentioning that 'full' second-order resources are already hugely controversial. Any significant weakening (*exempli gratia* to a theory with ancestral logic or a finiteness quantifier) usually results in the Downward Löwenheim-Skolem Theorem holding and hence the theory being unable to distinguish between V and a countable model.

[10] This follows from the *quasi*-categoricity of ZFC_2; we can only achieve (without hopelessly *ad hoc* anti-large cardinal axioms) isomorphism up to initial segments. *Id est* for any two models of ZFC_2, we know that they are isomorphic or one is isomorphic to a proper initial segment of the other, but cannot determine V up to isomorphism. Again, the literature here is extensive, but the reader is directed to [Isaacson, 2011] and [Meadows, 2013] for discussion.

[11] To see this, consider an object O_1 of fixed length moving (relative to a bystanding observer O_2) at a speed close to that of light past O_2. Further assume that O_1 has mirrors fitted to its the front and back that are able to reflect light pulses both back at O_1 and to O_2, given the emission of a pulse of light from an observer at the centre of O_1. Let the O_1-observer emit light pulses towards the mirrors as she passes O_2. Since the speed of light is constant for all observers under STR, the O_1-observer judges the light pulses hitting the front mirror as simultaneous with the light pulse hitting the back mirror from the perspective of her coordinate system. However, because the speed of light is constant for O_2, she will judge the light pulse to hit the rear mirror first, as the rear mirror has moved (from the perspective of her coordinate system) in the time it takes the light to travel.

[12] See [Bourne, 2006] for a discussion.

[13] See, [Bourne, 2006] for a clear exposition.

[14] See [Earman, 1995] and [Bourne, 2006] for the details.

particular events that are (relative to an observer) both future and past from the perspective of the reference frame, thereby preventing the partition of the model into the relevant absolute frames necessary to determine the truth of tensed facts.

The conclusion Gödel drew from this fact was that time must be ideal (*id est* 'unreal' in the McTaggartian sense). For, since $[RTT]$ is conceptually true of time (according to Gödel), we should expect every possible world in which time exists to admit of an interpretation on which there are absolute tensed facts. The Gödel metric shows the impossibility of such a situation[15]. Though the target of Gödel's argument is the unreality of time, it can equally be viewed as a *reductio* on the $[RTT]$-theorist's position; if one's theory concerning the (absolute!) nature of time leads to the unreality of time, then one is reduced to absurdity.

In order to see exactly where the $[RTT]$-theorist might find a response, we first lay out the argument in a clearer form. The following presentation is due to [Bourne, 2006] (p213):

> (P1) $[RTT]$ is necessary for the existence of time.
>
> (P2) If A is necessary for B, A exists in every possible world in which B exists.
>
> (3) For time to exist $[RTT]$ must be true in every possible world where time exists [by (P1) and (P2)]
>
> (P4) There is a world in which $[RTT]$ is false [as witnessed by the Gödel metric].
>
> (C5) Time cannot exist [from (3) and (P4)].

We can now see the direct affinity with the Universist's problem in Set Theory. Gödel's argument against the reality of time and tensed facts identifies claims about necessity (*id est* (P1) and (P2)), combines this with a possibility claim (P4), and then uses this to pull the claim back to the actual world ((C5)). Similarly, the Multiversist uses a principle about the *necessity* of mathematical objects, in conjunction with a claim about the *possibility* of an object (namely [Simulation]), and uses this to draw claims about the actual world, *videlicet* that the 'extra-V' entities actually exist. It is, therefore, instructive to consider the analogy between the two arguments.

4.2 The $[RTT]$-Theorist's Response.

How then does the Tense Theorist respond to Gödel's argument? (P2) seems fairly secure, as does the inference from (P1) and (P2) to (3). Similarly, given (3) and (P4) the inference to the conclusion looks like a simple application of modal logic. This leaves us with two main forms of attack. First we may challenge (P4) and argue that the Gödel metric does not represent a possibility of the desired kind. Second, we might scrutinise (P1); the claim that tense is necessary for the existence of time.

(P1) certainly *seems* initially plausible, if time is to exist at all, and one is an $[RTT]$-theorist about time, we should be able to speak of events as being determinately past, present, or future. Assuming that $[RTT]$ is conceptually true of time, it should hold of time in all possible worlds.

However, we should make a distinction here between what is referred to by 't' in the EFEs, and what *we* refer to by the term '*time*'. It is a platitudinous point, but under $[RTT]$ whatever *we* refer to when using the term 'time' *is* tensed; the making of tensed claims is essential to our navigating of the world and we regard ourselves

[15] There are a wealth of further solutions. For a survey, see [Earman, 1995].

as present (even if we can never *observe* the simultaneity of other evens). It turns out that (assuming $[RTT]$), the EFEs and use of the term 't' to co-refer to what we mean by 'time' provide a sophisticated and rigorous way of predicting phenomena within our universe. The Gödel metric shows that one can provide an interpretation of the EFEs on which 't' does *not* refer to something that can be partitioned into absolutely tensed planes.

However, we might dispute that 't' in the Gödel metric really does refer to 'time' as we understand it. The situation postulated is one on which it is possible (given some rather fantastical kit[16] to go into the future and arrive in the past. The physical possibility of such a situation is *highly* controversial, not least because of the host of conceptual problems associated with causal loops.

It is entirely possible then for the $[RTT]$-theorist to *deny* that the Gödel metric *does* in fact show the genuine physical possibility of a world on which $[RTT]$ is false, thus denying (P4). We are then able to maintain the truth of (P1), $[RTT]$-continues to be necessary on the conception of modality as physical possibility.

However, it is hard to resist the claim that *something* is shown by the Gödel metric. The $[RTT]$-theorist can reconcile this by noting that (P4) can be maintained if we *relax* the conception of modality in play. By moving to a *mathematical* modality, on which we view a model $\mathfrak{M}$ as possible relative to some theory T just in case $\mathfrak{M}$ does not satisfy the negation of one of the sentences of T, we can satisfy (P4). For the Godel metric provides a mathematical witness to the falsity of $[RTT]$ in a particular world. However, there is then little pressure to accept the truth of (P1), $[RTT]$ holds of whatever we denote by the term 'time', which need not be whatever is denoted by the term 't' in the EFEs.

Thus we see a possible resolution of the problem. For (P4) to be true, we have to be interpreting time to mean 'whatever it is that is denoted by t in the EFEs'. This allows us to interpret 'time' very generously, allowing in plausibly mathematical but non-physical possibilities. For (P1) to be true, we need that 'time' refer to whatever *we* refer to by our use of the term 'time'. It is not clear that this is amenable to interpretation as 't' in mathematical models on which basic intuitions concerning causation are violated.

5 Evaluating the Modal Argument.

This gives us the resources to respond to the Multiversist's problem against the Universist. Recall the two central premises up for consideration:

> [Necessity] Mathematical objects exist out of necessity if at all.
>
> [Simulation] If a view A accepts some entities xx, and it is possible to provide a model-theoretic interpretation of a theory T using the xx (in the sense of being able to satisfy the claims of T with an interpretation that makes use of the xx) then A should accept the possibility of the objects that we attempt to refer to with T.

I argue that [Necessity] is a close analogue of (P1) in Gödel's argument, and similarly [Simulation] is a close analogue of (P4). To see this consider the kinds of theories which we may model-theoretically simulate from within V on a Universist's perspective. It has been known since Paul Bernays 1945 proof of the independence of the Axiom of Foundation from $[ZFC -$ Foundation$]$ that given that V satisfies

[16] *Id est* a very fast rocket ship and a way of keeping one alive for a *very* protracted length of time.

[$ZFC-$ Foundation] that we can model-theoretically simulate the existence of non-well-founded sets. These are entities which the Universe theorist is likely to reject, especially given an adherence to the Iterative Conception.

This is indicative of a problem at the heart of [Simulation]; it simply includes *far* too much. Model-theoretically, it is possible to give a huge variety of models for wild and wonderful structures. The problem is easily pushed to absurd levels. For, it is easy to provide model-theoretic interpretations of paraconsistent and dialethic structures in models of ZFC (see [Friend, 2014] for some details and a historical catalogue). It would be far beyond what the Universist would have ever countenanced to admit the even *metaphysical possibility* of such structures (the dialethic case is particularly vivid). [Simulation] then is *obviously false* given the understanding of the modality as metaphysical, just as (P4) was false for the $[RTT]$-theorist given the understanding of modality as *physical* possibility there.

It is hard, however, to resist the claim that *something* is shown by the model-theoretic interpretations. Study of this helps us to diagnose exactly what has gone wrong in the Multiversist argument against the Universist. We can see the argument through analogy with the case of the $[RTT]$-theorist from earlier. There it was noted that both (P1) and (P4) were true for the $[RTT]$-theorist, but only given divergent interpretations of the modality. A similar phenomenon has occurred here. For [Necessity] is clearly, on the Universist's picture, a claim about *metaphysical* possibility. Any mathematical object that could *metaphysically* exist, *does* exist. However, the simple examples from non-well-founded and paraconsistent models show that the Universist is unlikely to accept that [Simulation] is true where the modality is interpreted *metaphysically*.

We can, however, rehabilitate [Simulation] if the modality is substantially weakened. If interpreted in a merely platitudinous sense; these sorts of objects *are* possible, where 'possible' is simply taken to mean 'does not conflict with a particular formal system'. The modality is perfectly legitimate, it is just not the salient kind for bringing the objects back down to the actual world using [Necessity].

Return then to the case of forcing and proper classes. The Universist may agree that something is shown by the relevant constructions, the possibility of such objects in the sense that their existence does not conflict with the axioms of ZFC *modulo* first-order logic. However, it is incorrect to say that this establishes their possibility in a more substantive metaphysical sense. Indeed, it is clear that the Universist holds that mathematical reality is more highly constrained than mere model-theoretic simulation from the get go, as evidenced by her attitude to the simple cases of non-well-founded and paraconsistent models. The Universist is forced to accept that mathematical reality is more highly constrained than what can be model-theoretically simulated in a first-order setting. But this was *always* a part of her view (which, as noted in §1, she already has some reason to accept), and the simulations of proper classes and forcing extensions do nothing to further acceptance of [Simulation].

One further question remains for the Universist; why accept [Simulation] in the case of the Complex Numbers? I make the following bold conjecture; the Universist need not accept [Simulation] as the warrant for accepting $\mathbb{C}$ as a legitimate entity. One must ask, given the acceptance of $\mathbb{C}$, "what was the warrant for doing so?". Was it the fact that [Simulation] is satisfied by the interpretation of $\mathbb{C}$ in $\mathbb{R}$? Or was it rather that the representation of $\mathbb{C}$ as a number-plane rather than a number-line provided an independent picture on which it could be seen that a coherent structure was thereby defined? It should be noted that there is *no* pressure, once the understanding *via* a plane had been given, that any complex number with an imaginary part was *included* in the real line. However, in virtue of the conception of V held by the Universist, namely that V constitutes *all* sets, there is such a pressure to assert that the 'extra-V' objects simulated are already in V. The reason [Simulation] *seems* warranted in the case of $\mathbb{C}$ is that it is a happy mathematical coincidence that the simulation points the way to an independent conception of the structure in question. In the cases of proper classes and forcing extensions, no such independent conception is available.

Conclusion.

Let us take stock. We have seen that the arguments from proper classes and forcing extensions appear at first sight to present a new problems of Naturalism for the Universist. However, the Naturalistic costs have been reduced in virtue of attempts to code the purported new objects as closely as possible using objects in V. Given this, an argument to strengthen the problem is desirable. One option is to argue for a deeper problem based on modal principles held by the Universist. However a deeper analysis of the argument, precipitated by analogical features with Gödel's arguments against the reality of time, show the modal objection to depend on an equivocation between distinct varieties of modality. Examination of more basic cases indicates that the Universist is likely to have already rejected the modal argument on far simpler grounds. In this way, the modal argument fails to deepen the Naturalistic problems facing the Universist.

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