# Varieties of concept algebras

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Concept algebras ($\mathsf{CA}$s) are central tools in connecting, combining and comparing theories. They are useful in investigating relativity theory as a network of logic theories as in [3]. The elements of the concept algebra of a given theory are the *concepts* definable in the theory, and the operations are the logical connectives that are used to form new concepts. Homomorphisms between concept algebras correspond to interpretations between theories, and equations valid in concept algebras correspond to formula-schemes like, e.g., $\varphi \to \exists v_i \varphi$. Concept algebras of first-order logic were named *cylindric algebras* by Alfred Tarski, because of the geometrical meaning of the quantifiers. When we use $\alpha$ many variables $v_0, \dots, v_i, \dots (i \in \alpha)$ the concepts are subsets of an $\alpha$-dimensional Cartesian space, and $\mathsf{CA}_\alpha$ denotes the class of $\alpha$-dimensional cylindric algebras. Concept algebras synthesize logic, algebra and geometry.

Problem 4.2 in Henkin-Monk-Tarski [2] asks for the number of varieties (equationally defined subclasses) of $\mathsf{RCA}_\alpha$, where the latter denotes the class of so-called representable (also called *geometrical* in the introduction of [1]) concept algebras. It was known that there are at least continuum many for infinite $\alpha$, and the problem asks whether there are more in case $\alpha > 2^\omega$.

**Theorem 1** (solution of [2, Problem 4.2]) *There are $2^\alpha$ many subvarieties of $\mathsf{RCA}_\alpha$, for $\alpha \geq \omega$.*

The key step in proving Theorem 1 is to construct an $\mathsf{RCA}_\alpha$ and an equation $e$ such that $e$ fails in the algebra, but all the versions of $e$ where we rename the indices occurring in $e$ (any way) hold in the algebra. Let us call an $\mathfrak{A} \in \mathsf{CA}_\alpha$ *symmetric* if it is not like our counterexample, i.e., if $\mathfrak{A} \models \rho(e)$ whenever $\mathfrak{A} \models e$ and $\rho : \alpha \to \alpha$ is a one-one function for renaming the indices occurring in $e$. We found that, surprisingly, almost all $\mathsf{CA}_\alpha$s are symmetric (see the theorem below). Clearly, there are only continuum many varieties of symmetric $\mathsf{CA}_\alpha$s, for any $\alpha$, because the equations using indices from $\omega$ only determine such an equational theory.

Let's call an algebra $\mathfrak{A} \in \mathsf{CA}_\alpha$ *endo-dimension complemented*, endo-dc for short, if for any $\Gamma \subseteq \alpha$ and nonzero $a \in A$ there are an endomorphism of the $\Gamma$-reduct of $\mathfrak{A}$ and a $\kappa \in \alpha$ such that $h(a)$ is still nonzero and each element of the range of $h$ is $\kappa$-closed. Let $\mathsf{Lf}_\alpha, \mathsf{Dc}_\alpha, \mathsf{Di}_\alpha, \mathsf{Edc}_\alpha$ denote the classes of all locally finite-dimensional, dimension-complemented, diagonal, and endo-dc $\mathsf{CA}_\alpha$s respectively. It is proved in [1, 2.6.50] that $\mathsf{Lf}_\alpha \subset \mathsf{Dc}_\alpha \subset \mathsf{Di}_\alpha \subset \mathsf{Edc}_\alpha \subseteq \mathsf{RCA}_\alpha$, and it was asked as [1, Problem 2.13] whether the last inclusion is strict or not. We proved that all endo-dc algebras are symmetric, so almost all interesting $\mathsf{CA}_\alpha$s are symmetric. Also, the algebra we used in the proof of Theorem 1 is representable but not symmetric, hence solves this problem in the negative.

**Theorem 2** (solution of [1, Problem 2.13])

*(i) All endo-dc algebras are symmetric.*

*(ii) There is a representable* $\mathsf{CA}_\alpha$ *which is not endo-dc, but each* $\mathsf{RCA}_\alpha$ *can be embedded into an endo-dc algebra.*

Symmetric algebras are not necessarily representable, so there is a big gap between endo-dc and symmetric algebras in this sense. We found a class of algebras which sits just at the right place between endo-dc and symmetric algebras. Let us call an $\mathfrak{A} \in \mathsf{CA}_\alpha$ *lifting* if $\mathfrak{A} \models e$ whenever $\mathfrak{A} \models e(c_i x)$ and $i$ does not occur in $e$.

**Theorem 3**

*(i) Each endo-dc algebra is lifting and each lifting algebra is symmetric and representable.*

*(ii) The inclusions in* $(i)$ *above are strict: there is a lifting algebra which is not endo-dc and there is a symmetric* $\mathsf{RCA}_\alpha$ *which is not lifting.*

Moreover, we found that lifting algebras are in intimate connection with $\mathsf{Lf}_\alpha$'s (and thus with $\mathsf{Dc}_\alpha$s, $\mathsf{Di}_\alpha$s, $\mathsf{Edc}_\alpha$s) in the following sense. Let us call a set $E$ of equations lifting iff $e \in E$ whenever $e(c_i x) \in E$ for some $i$ not occurring in $e$. In this case we also say that $E$ is closed under the *lifting rule*.

**Theorem 4**

*(i) An algebra is lifting iff it is equationally indistinguishable from some* $\mathsf{Lf}_\alpha$.

*(ii) A semantically closed set $E$ of equations is lifting iff it is the equational theory of some class of locally finite-dimensional algebras.*

As a corollary of the above theorem, we get a new enumeration/description of the equational theory of $\mathsf{RCA}_\alpha$, radically simpler than those that can be found in [2, pp.112-119]. This may contribute to the solution of [2, Problem 4.1] which asks for a simple equational basis for the equations valid in $\mathsf{RCA}_\alpha$.

**Theorem 5** *The equational theory of* $\mathsf{RCA}_\alpha$ *is the smallest set of equations containing the cylindric axioms* $(C0)-(C7)$*, closed under the rules of equational logic, and closed under the lifting rule.*

The above theorem also can contribute to giving solutions to [2, Problem 4.16], different from the ones in [5, 6].

# References


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