Square Paths and Cycles

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(Joint work with Phong Châu and Louis DeBiasio)

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June 2011
Thanks

Thank you Professor Hajnal, and Professors Erdős and Szemerédi, for extremal graph theory!
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Thanks

Trotter, Szemerédi, Genghua Fan, Häggkvist, Sárközy, Czygrinow, Katona^2, Kostochka, Rućinski
Theorem (Dirac (1952))

A graph $G$ has a Hamiltonian cycle if

$$\delta(G) \geq \frac{1}{2} |G| > 1.$$

Example 1.

1. $K_2$
2. $G' = K_t + 1 \lor K_t$:
   $$\delta(G') = t < t + 1 \frac{1}{2} = \frac{1}{2} |G'|$$
3. $G''$: $G'' = (K_t + K_t) \lor K_1$:
   $$\delta(G'') = t < t + 1 \frac{1}{2} = \frac{1}{2} |G''|$$

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1952—Hamiltonian cycles

Theorem (Dirac (1952))

$G$ has a hamiltonian cycle if $\delta(G) \geq \frac{1}{2} |G| > 1$.

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Theorem (Dirac (1952)\(^1\))

*G* has a *hamiltonian cycle* if \(\delta(G) \geq \frac{1}{2}|G| > 1.\)

**Example**

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$G$ has a hamiltonian cycle if $\delta(G) \geq \frac{1}{2}|G| > 1$.

Example

1. $K_2$
2. $G := K_{t+1} \lor K_t$: $\delta(G) = t < t + \frac{1}{2} = \frac{1}{2}|G|$

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Theorem (Dirac (1952)\(^1\))

*G* has a hamiltonian cycle if \(\delta(G) \geq \frac{1}{2}|G| > 1\).

**Example**

1. \(K_2\)
2. \(G := \overline{K_{t+1}} \vee K_{t}:: \delta(G) = t < t + \frac{1}{2} = \frac{1}{2}|G|\)
3. \(G := (K_t + K_t) \vee K_1:: \delta(G) = t < t + \frac{1}{2} = \frac{1}{2}|G|\)

Proof of Dirac’s Theorem

Let $\delta(G) \geq \frac{1}{2} |G|$. Consider a maximum path.
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Let $\delta(G) \geq \frac{1}{2}|G|$. Consider a maximum path.

The green cycle must be hamiltonian!
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The green cycle must be hamiltonian!
Theorem (Ore [1960]$^2$)

$G$ is hamiltonian if for all distinct $x, y \in V$ with $xy \notin E$

$$d(x) + d(y) \geq |G|.$$
Theorem (Corrádi & Hajnal\textsuperscript{3})

Let $G$ be a graph on $n$ vertices with $\delta(G) \geq 2k$. Then $G$ contains $k$ disjoint cycles.

Corollary

If $G$ is a graph $n = 3k$ vertices with $\delta(G) \geq 2k = \frac{2}{3}n$ then $V(G)$ can be partitioned into 3-cliques.

Theorem (Hong Wang (2010))

If $G$ is a graph $n = 4k$ vertices with $\delta(G) \geq 2k = \frac{1}{2}n$ then $V(G)$ can be partitioned into 4-cycles.

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Definition

A square cycle is a cycle together with every 2-chord.
1964—Pósa’s Conjecture

Conjecture (Pósa (1964)\textsuperscript{5})

*Every graph $G$ with minimum degree $\delta(G) \geq \frac{2}{3}|G|$ has a hamiltonian square cycle.*

Conjecture (Pósa (1964)\textsuperscript{5})

*Every graph $G$ with minimum degree $\delta(G) \geq \frac{2}{3}|G|$ has a Hamiltonian square cycle.*

**Example**

$G := K_{2t-1} \lor \overline{K}_t$: $\delta(G) = 2t - 1 < 2t - \frac{2}{3} = \frac{2}{3}(3t - 1) = \frac{2}{3}|G|$ and it has no Hamiltonian square cycle.

1964—Equitable coloring

Definition
An equitable \( r \)-coloring of a graph \( G \) is a proper \( r \)-coloring, for which any two color classes differ in size by at most one.

\[ |G| = rs \]
Conjecture (Erdős [1964]6)

Every graph $G$ with $\Delta(G) \leq r$ has an equitable $(r + 1)$-coloring.
Conjecture (Erdős [1964]⁶)

Every graph $G$ with $\Delta(G) \leq r$ has an equitable $(r + 1)$-coloring.

Example

If $|G| = 3(r + 1)$ and $\Delta(G) \leq r$ then $\delta(\overline{G}) \geq 2(r + 1)$. By Corrádi-Hajnal $\overline{G}$ is spanned by $r + 1$ triangles, i.e., $G$ has an equitable $(r + 1)$-coloring.

1970—Erdős’ Conjecture is true

Theorem (Hajnal & Szemerédi [1970]⁷)

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Theorem (Hajnal & Szemerédi [1970])

Every graph $G$ with $\Delta(G) \leq r$ has an equitable $(r + 1)$-coloring.

- The proof does not yield a polynomial time algorithm.

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The $k$-power of a cycle is a cycle together with all $i$-chords, $2 \leq i \leq k$
1973—Powers of Cycles

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Conjecture (Seymour [1973])
Every graph \( H \) with minimum degree \( \delta(H) \geq \frac{s}{s+1} |H| \) contains the \( s \)-power of a hamiltonian cycle.
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- The case \( s = 1 \) is Dirac’s Theorem.
- The case \( s = 2 \) is Pósa’s Conjecture.
Motivation for Seymour’s Conjecture

Let $|G| = (r + 1)(s + 1)$. Then

\[ \Delta(G) \leq r \text{ iff } \delta(G) \geq |G| - (r + 1) = \frac{s}{s+1} |G|; \quad \text{and} \]

if $C$ is the $s$-power of a hamiltonian cycle in $G$, then all sets of $s + 1$ consecutive vertices of $C$ are cliques in $G$ and independent sets in $G$. Thus $G$ has an equitable $(r + 1)$-coloring.
Motivation for Seymour’s Conjecture

Let $|G| = (r + 1)(s + 1)$. Then

- $\Delta(G) \leq r$ iff $\delta(\overline{G}) \geq |G| - (r + 1) = \frac{s}{s+1} |\overline{G}|$; and
- if $C$ is the $s$-power of a Hamiltonian cycle in $\overline{G}$ then all sets of $s + 1$ consecutive vertices of $C$ are cliques in $\overline{G}$ and independent sets in $G$. Thus $G$ has an equitable $(r + 1)$-coloring.
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\[\overline{G}\]
Theorem (Fan & Kierstead(1995)$^8$)
\[ \forall \varepsilon > 0 \ \exists m \text{ s.t. if } \delta(G) \geq \left(\frac{2}{3} + \varepsilon\right)|G| + m \text{ then all } wx, yz \in E \text{ have a }\textit{hamiltonian} \text{ square } wx, yz\text{-path } wx \ldots yz.\]

Theorem (Fan & Kierstead(1995)\textsuperscript{8})
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Definition
A square \( wx, yz \)-path is \textbf{optimal} if among maximum such paths, it has as many 3-chords, and then as many 4-chords as possible.

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Lemma (Optimal Path)
\textit{Let } \(C\) \textit{ be an optimal } \(wx, yz\)-\textit{path and } \(H := G - C\). Then
\[ \|y, C\| \leq \frac{2}{3}|C| + 1 \text{ for all } y \in H. \]

Lemma (Connecting)

\[ \delta(G) > \frac{2}{3} |G| \implies \text{all } wx, yz \in E \text{ have a square } wx, yz\text{-path.} \]
Reservoirs

Let \( G \) be a graph and \( 0 < \rho < 1 \).
Reservoirs

- Let $G$ be a graph and $0 < \rho < 1$.
- We would like to find a subgraph $R \subset G$ with $|R| = \rho |G|$ that “looks like” $G$.

![Diagram of a graph $G$ with a subgraph $R$]
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- “Ideally” $|S \cap R| = \rho|S|$ for every $S \subset V(G)$.
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- “Ideally” $|S \cap R| = \rho|S|$ for every $S \subset V(G)$.
- We can not achieve this for all $S$, but using Chernoff’s bound we can achieve $||S \cap R| - \rho|S|| < \gamma|G|$ for $|G|^c$ subsets $S$. 

![Diagram of a graph G with a subgraph R and a subset S]
Let $G$ be a graph and $0 < \rho < 1$.

We would like to find a subgraph $R \subset G$ with $|R| = \rho |G|$ that “looks like” $G$.

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We cannot achieve this for all $S$, but using Chernoff’s bound we can achieve $\| |S \cap R| - \rho |S| \| < \gamma |G|$ for $|G|^c$ subsets $S$.

Which $S$ do we need?
Reservoir Lemma

Lemma (Reservoir)

If $|G|$ is sufficiently large then $\forall \gamma, \rho > 0 \exists R \subseteq G \ \forall x \in V \ s.t.

|R| = \rho |G| \ and \ |N(x) \cap R| - \rho d(x)| < \gamma |G|

.$
Theorem (G. Fan & Kierstead(1995))

\[ \forall \varepsilon > 0 \ \exists m \text{ s.t. if } \delta(G) \geq \left( \frac{2}{3} + \varepsilon \right) |G| + m \text{ then all } wx, yz \in E \text{ have a Hamiltonian } wx, yz\text{-square path } wx \ldots yz. \]

Proof.
Induction on $|G|$.

\[ \square \]

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Sketch of Proof

Theorem (G. Fan & Kierstead(1995)\textsuperscript{9})
\[\forall \varepsilon > 0 \ \exists m \text{ s.t. if } \delta(G) \geq \left(\frac{2}{3} + \varepsilon\right) |G| + m \text{ then all } wx, yz \in E \text{ have a hamiltonian } wx, yz\text{-square path } wx \ldots yz.\]

Proof.
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Sketch of Proof

Theorem (G. Fan & Kierstead(1995))

∀ε > 0 ∃m s.t. if δ(G) ≥ (2/3 + ε)|G| + m then all wx, yz ∈ E have a hamiltonian wx, yz-square path wx…yz.

Proof.
Induction on |G|.

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Induction on $|G|$.

1995–1998—Pósa’s and Seymour’s Conjectures

Theorem (G. Fan & Kierstead (1996)\textsuperscript{10})

Every graph $G$ with $\delta(G) \geq \frac{2|G|−1}{3}$ has a hamiltonian square path.


\textsuperscript{12}Partitioning a graph into two square-cycles. J. Graph Theory 23 (1996), no. 3, 241–256.
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Every graph $G$ with $\delta(G) \geq \frac{2|G|-1}{3}$ has a hamiltonian square path.

Corollary (Aigner-Brandt Theorem\textsuperscript{11})

If $\delta(G) \geq \frac{2|G|-1}{3}$ then $H \subseteq |G|$ for all $H$ with $|H| \leq |G|$ and $\Delta(H) \leq 2$.


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Theorem (G. Fan & Kierstead (1996)\textsuperscript{12})

Let $G$ be a graph with $\delta(G) \geq \frac{2}{3}|G|$. If graph $G$ has a square cycle of length greater than $\frac{2}{3}|G|$ then $G$ has a hamiltonian square cycle; otherwise $V(G)$ can be partitioned into two square cycles.


\textsuperscript{12}Partitioning a graph into two square-cycles. J. Graph Theory 23 (1996), no. 3, 241–256.
Theorem (Komlós & Sárközy & Szemerédi (1996)$^{13}$)

There exists $n \in \mathbb{N}$ such that every graph $G$ with $|G| \geq n$ and $
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1995–1998—Results on Pósa and Seymour’s Conjectures

Theorem (Komlós & Sárközy & Szemerédi (1996)\textsuperscript{13})

There exists \( n \in \mathbb{N} \) such that every graph \( G \) with \( |G| \geq n \) and \( \delta(G) \geq \frac{2}{3} |G| \) contains a hamiltonian square cycle.

Key idea: Let \( \alpha > 0 \). Consider whether or not:

\[ \exists A, B \subset V, \left( \frac{1}{3} - \alpha \right) \leq |A|, |B| \leq \frac{1}{3} |G| \land \|A, B\| \leq \alpha |A||B|. \]

- If yes (extreme case) then prove directly.
- Else (nonextreme case) use the Regularity-Blow-Up method.

1995–1998—Pósa’s and Seymour’s Conjectures

Theorem (Komlós & Sárközy & Szemerédi (1998)\textsuperscript{14})

For every $s \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that every graph $G$ with $|G| \geq n$ and $\delta(G) \geq \frac{s}{s+1} |G|$ contains the $s$-power of a hamiltonian cycle.

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Theorem (Komlós & Sárközy & Szemerédi (1998)\textsuperscript{14})

For every \( s \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that every graph \( G \) with \( |G| \geq n \) and \( \delta(G) \geq \frac{s}{s+1} |G| \) contains the \( s \)-power of a Hamiltonian cycle.

- The KSS proof uses the Hajnal-Szemerédi Theorem, Szemerédi’s Regularity Lemma and their Blow-Up Lemma.
- It does not yield an simple proof of the Hajnal-Szemerédi Theorem.

Theorem (Kierstead & Kostochka & Mydlarz & Szemerédi (2010)).

Every graph on $n$ vertices with maximum degree at most $r$ can be equitably $(r+1)$-colored in $O(n^2)$ steps.

\[ \text{A fast algorithm for equitable coloring, Combinatorica, 30 (2010) 217–224} \]
Theorem (Kierstead & Kostochka & Mydlarz & Szemerédi (2010).\textsuperscript{15})

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\textsuperscript{15}A fast algorithm for equitable coloring, *Combinatorica*, 30 (2010) 217–224
An Ore-type version of equitable coloring

Theorem (Kierstead & Kostochka (2008))

Every graph satisfying \( d(x) + d(y) \leq 2r + 1 \) for every edge \( xy \), has an equitable \((r+1)\)-coloring.

The proof does not provide a polynomial time algorithm.

Problem

Find a polynomial time algorithm for the coloring established by the theorem above.

An Ore-type version of equitable coloring

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Find a polynomial time algorithm for the coloring established by the theorem above.

Conjecture (Kierstead)

If \( d(x) + d(y) \geq \frac{4}{3} |G| \) for all \( xy \notin E \) then \( G \) has a hamiltonian square cycle.
An Ore-type Pósa Example (Châu)

$|G| = 30$

$d(x) + d(y) \geq 40, \forall xy \notin E$

$K_{17}$
An Ore-type Pósa theorem

Let graph $G$ satisfy
$$d(x) + d(y) \geq \frac{4}{3}|G| - \frac{1}{3}$$
for all $xy \not\in E$.

If
$$\frac{1}{3}|G| + \frac{5}{3} \leq \delta(G) \leq \frac{1}{3}|G| + \frac{2}{3},$$
then $G$ has a hamiltonian square path.

If
$$\delta(G) > \frac{1}{3}n + \frac{2}{3},$$
and $n$ is sufficiently large, then $G$ contains a hamiltonian square cycle.

\^17Ph.D. thesis, Arizona State University
An Ore-type Pósa theorem

Theorem (Chau (2009)\textsuperscript{17})

Let graph $G$ satisfy $d(x) + d(y) \geq \frac{4}{3}|G| - \frac{1}{3}$ for all $xy \notin E$.

- If $\frac{1}{3}|G| + \frac{5}{3} \leq \delta(G) \leq \frac{1}{3}|G| + 2$ then $G$ has a hamiltonian square path.

- If $\delta(G) > \frac{1}{3}n + 2$, and $n$ is sufficiently large, then $G$ contains a hamiltonian square cycle.

\textsuperscript{17}Ph.D. thesis, Arizona State University
The Ore-type Aigner-Brandt theorem

Theorem (Kostochka & G. Yu\textsuperscript{18})

If graph $G$ satisfies $d(x) + d(y) \geq \frac{4}{3}|G| - 1$ for all $xy \notin E(G)$ then $H \subseteq G$ for all graphs $H$ with $|H| \leq |G|$ and $\Delta(H) \leq 2$. 

\textsuperscript{18}Ore-type conditions implying 2-factors consisting of short cycles. Discrete Math., in press
Hypergraphs and chains

Definition
Let $H = (V, E)$ be an $s$-uniform hypergraph ($s$-graph).

- The co-degree: For an $(s - 1)$-set $\bar{x} \subseteq V$:
  
  $$d_{s-1}(\bar{x}) = |\{ e \in E : \bar{x} \subset e \}|; \quad \delta_{s-1}(H) = \min_{\bar{x} \subseteq V} d_{s-1}(\bar{x})$$

- A closed chain is a sequence

  $$x_1 x_2 \ldots x_{r+1} \ldots x_{r+s} \ldots x_t x_1 \ldots x_{s-1}$$

  such that $\{x_{r+1} \ldots x_{r+s}\} \in E$ for all $r \leq t - 1$.

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19 J Graph Theory 30: 205–212, 1999
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- **A closed chain** is a sequence

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  such that $\{x_{r+1} \ldots x_{r+s}\} \in E$ for all $r \leq t - 1$.

Proposition (Gy. Y. Katona and Kierstead (1999)\textsuperscript{19})

If $\delta_{s-1}(H) \geq b \coloneqq (1 - \frac{1}{2s})|H|$ then $H$ has a **closed hamiltonian chain**. There are counter-examples if $b \coloneqq \left\lfloor \frac{|H| - s + 1}{2} \right\rfloor$.

\textsuperscript{19}J Graph Theory 30: 205–212, 1999
Dirac-type chain theorem

Theorem (Rödl, Ruciński, and Szemerédi (2008) \textsuperscript{20})

For all positive integers $s$, $s$-graphs $H$ and $\gamma > 0$, if
\[
\delta_{s-1}(H) \geq (1/2 + \gamma)|H|\quad \text{and}\quad |H| \text{ is sufficiently large then } H \text{ has a hamiltonian closed chain.}
\]

\textsuperscript{20}An approximate Dirac-type theorem for $k$-uniform hypergraphs,

\textsuperscript{21}Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs, manuscript
Dirac-type chain theorem

Theorem (Rödl, Ruciński, and Szemerédi (2008)\textsuperscript{20})

For all positive integers $s$, $s$-graphs $H$ and $\gamma > 0$, if $\delta_{s-1}(H) \geq (1/2 + \gamma)|H|$ and $|H|$ is sufficiently large then $H$ has a hamiltonian closed chain.

Theorem (Rödl, Ruciński, and Szemerédi (2010)\textsuperscript{21})

Let $H$ be a 3-graph with $\delta_2(H) \geq \lfloor |H|/2 \rfloor$ and $|H|$ sufficiently large. Then $H$ has a hamiltonian closed chain. This is tight.

\textsuperscript{20}An approximate Dirac-type theorem for $k$-uniform hypergraphs, Combinatorica 28 (2) (2008) 229-260.

\textsuperscript{21}Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs, manuscript
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Theorem (Rödl, Ruciński, and Szemerédi (2010)\textsuperscript{21})

Let $H$ be a 3-graph with $\delta_2(H) \geq \left\lfloor |H|/2 \right\rfloor$ and $|H|$ sufficiently large. Then $H$ has a hamiltonian closed chain. This is tight.

Idea: Consider an extremal and nonextremal case. For the latter, construct a short $\varepsilon$-absorbing chain, extend to a $(1 - \varepsilon)|H|$-closed chain, absorb remaining vertices.

\textsuperscript{20}An approximate Dirac-type theorem for k-uniform hypergraphs, Combinatorica 28 (2) (2008) 229-260.

\textsuperscript{21}Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs, manuscript
Theorem (Levitt, Sárközy and Szemerédi (2010)\textsuperscript{22})

Pósa’s Conjecture is true for graphs with at least $10^C$ vertices.

\textsuperscript{22}How to avoid using the Regularity Lemma: Pósa’s conjecture revisited, Discrete Mathematics 310 (2010) 630–641.

\textsuperscript{23}Pósa’s Conjecture for graphs of order at least $2 \times 10^8$, Random Structures and Algorithms, to appear.
Theorem (Levitt, Sárközy and Szemerédi (2010))

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The proof does not depend on the Regularity and Blow-Up Lemmas, and uses Reservoirs.

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$^{23}$Pósa’s Conjecture for graphs of order at least $2 \times 10^8$, Random Structures and Algorithms, to appear.
2010—Toward a proof of Pósa’s Conjecture

Theorem (Levitt, Sárközy and Szemerédi (2010)\textsuperscript{22})

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Theorem (Châu, DeBiasio and Kierstead\textsuperscript{23})

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\textsuperscript{22} How to avoid using the Regularity Lemma: Pósa’s conjecture revisited, Discrete Mathematics 310 (2010) 630–641.

\textsuperscript{23} Pósa’s Conjecture for graphs of order at least $2 \times 10^8$, Random Structures and Algorithms, to appear.
Definition
$S \subseteq V$ is $\alpha$-extreme if $\forall v \in S$, $|S| \geq (1 - \alpha) \frac{n}{3}$ and $\|v, S\| < \alpha \frac{n}{3}$.

Lemma (Extreme Case)
If $G$ has an $\frac{1}{36}$-extreme set then $G$ has a hamiltonian square cycle.
Nonextreme Case

Lemma (Long Path Lemma)

If $\delta(H) \geq \left(\frac{2}{3} - \varepsilon\right)|H|$ then $H$ has disjoint square paths $P_1$ and $P_2$ s.t. $|P_1| + |P_2| > \left(\frac{5}{6} - 2\varepsilon\right)|H|$.
Nonextreme Case

Lemma (Long Path Lemma)
If $\delta(H) \geq (\frac{2}{3} - \varepsilon) |H|$ then $H$ has disjoint square paths $P_1$ and $P_2$ s.t. $|P_1| + |P_2| > (\frac{5}{6} - 2\varepsilon)|H|$.

Definition (Special Sets)
If $S = (N(u, v, w) \cup N(u, v, x)) \cap N(y)$ then $S$ is special.
Nonextreme Case

Lemma (Long Path Lemma)
If $\delta(H) \geq \left(\frac{2}{3} - \varepsilon\right)|H|$ then $H$ has disjoint square paths $P_1$ and $P_2$ s.t. $|P_1| + |P_2| > \left(\frac{5}{6} - 2\varepsilon\right)|H|$.

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If $S = (N(u, v, w) \cup N(u, v, x)) \cap N(y)$ then $S$ is special.

Lemma (Connecting Lemma)
[Connecting Lemma] Let $0 < \beta < \alpha \leq \frac{1}{64}$, $0 \leq \varepsilon \leq (\alpha - \beta)/16$, $l := 10$ and suppose $n \geq \max\left\{\frac{9l + 96}{\varepsilon}, \frac{3l + 39}{\beta}\right\}$. Suppose $H$ has no $\frac{1}{36}$-extreme special set and $\delta(H) \geq \left(\frac{2}{3} - \varepsilon\right)n$. $\forall ab, cd \in E$ there exists a square $ab, cd$-path $P$ with $|P| \leq 14$. 

Lemma (Reservoir Lemma)
Let $|H| \geq n^0 := 2 \times 10^8$ and $\delta(H) \geq \left(\frac{2}{3} - \varepsilon\right)|H|$. If $H$ contains no $\frac{1}{36}$-extreme set, then $H$ has a reservoir of size $\rho|H|$ that has no $\frac{1}{36}$-extreme special sets.
Nonextreme Case

Lemma (Long Path Lemma)
If $\delta(H) \geq \left(\frac{2}{3} - \varepsilon\right)|H|$ then $H$ has disjoint square paths $P_1$ and $P_2$ s.t. $|P_1| + |P_2| > \left(\frac{5}{6} - 2\varepsilon\right)|H|$.

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Completing the proof

1. Construct a special reservoir $R$. 

\[ G \]
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Completing the proof

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![Diagram showing the proof](image-url)
Completing the proof

1. Construct a special reservoir $R$.
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4. This gives a cycle of length greater than $\frac{2}{3}|G|$.

Diagram:

![Diagram showing a cycle connected through a reservoir](image)
Completing the proof

1. Construct a special reservoir $R$.
2. Construct two long paths in $H = G - R$.
3. Connect the paths through $R$.
4. This gives a cycle of length greater than $\frac{2}{3}|G|$.
5. Finish by Long Cycle Theorem.