On the eigenvectors of random regular graphs

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Random regular graphs

Fix $d \geq 3$.

$G = G(n, d)$: a **uniformly chosen**, simple d-**regular graph** on n labeled vertices (this is not the only possible model).

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eigenvalues: $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

eigenvectors: $v_1, \ldots, v_n$

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Friedman (2008), Bordenave (2015): $\lambda_2 \leq 2\sqrt{d-1} + \varepsilon$ with high probability

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How do the eigenvectors look like? $\Rightarrow$ geometric or clustering properties.

Distribution of the eigenvectors

A vector $v \in \mathbb{R}^n$ is an **eigenvector** with eigenvalue λ if

$$Av = \lambda v,$$

that is,

$$\sum_{(x,y)\in E} v(y) = \lambda v(x) \qquad \text{holds for all } x \in V.$$

We will always assume that $\|v\|_2 = 1$.

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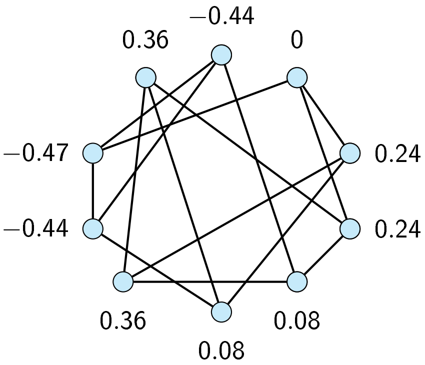
Does this probability distribution "converge" weakly as $n \to \infty$?

Example: second eigenvector

The following is the second eigenvector with $\lambda = 1.88$ ($n = 10, d = 3$):

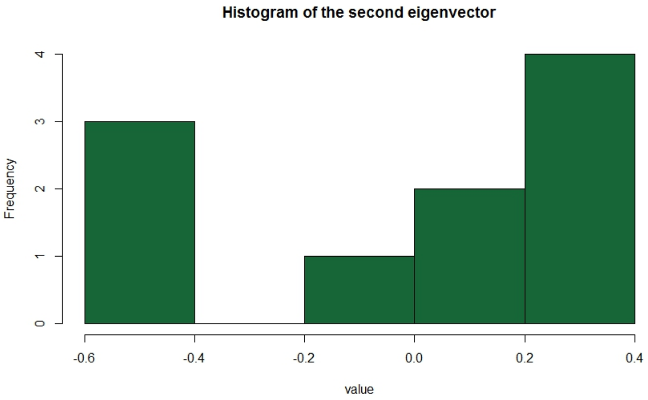


Example: histogram of the second eigenvector

Second eigenvector:

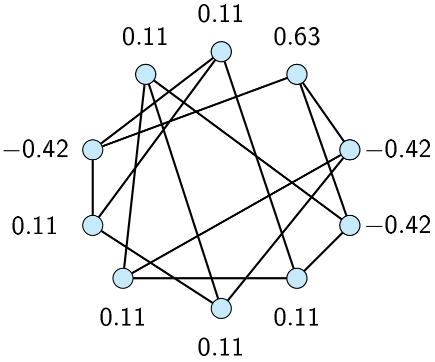
$$-0.47 \quad -0.44 \quad -0.44 \quad 0.00 \quad 0.08 \quad 0.08 \quad 0.24 \quad 0.24 \quad 0.36 \quad 0.36$$

The mean is 0, the standard deviation is 0.33.



Example: ninth eigenvector

The following is the ninth eigenvector with $\lambda = -2$ $(n = 10, d = 3)$:

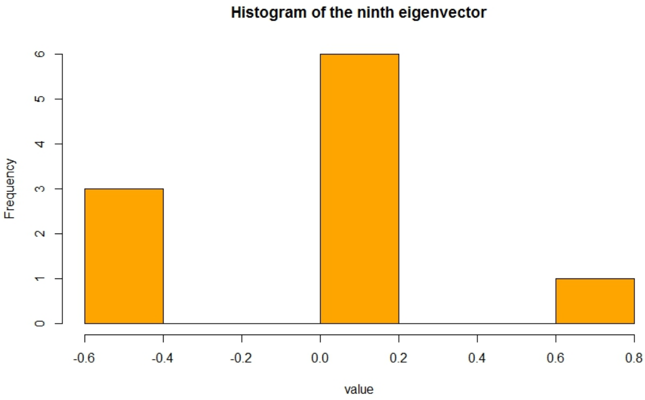


Example: histogram of the ninth eigenvector

Second eigenvector:

$$-0.42 \quad -0.42 \quad -0.42 \quad 0.11 \quad 0.11 \quad 0.11 \quad 0.11 \quad 0.11 \quad 0.11 \quad 0.63$$

The mean is 0, the standard deviation is 0.33.



Eigenvectors of random matrices

The distribution of a fixed coordinate of a fixed eigenvector (of length 1) multiplied by $\sqrt{n}$ tends to the **Gaussian distribution** for the following random matrices:

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- Wigner matrix: symmetric matrix with iid entries

  Tao–Vu (2012), Knowles–Yin (2013): Wigner matrix such that the first four moments of the entries are the same as the first four moments of the Gaussian distribution
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- Bourgade–Huang–Yau (2016+): Erdős–Rényi random graph with the average degree tending to infinity

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Fix a metrization of the weak topology (e.g. the Lévy–Prokhorov distance).

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Theorem (B–Szegedy, 2016+)

For all $\varepsilon > 0$ there exists N such that for every $n \geq N$ the following holds for a random d-regular graph $G(n, d)$ with probability at least $1 - \varepsilon$.

For every non-constant eigenvector v of G (with $\|v\|_2 = 1$) there exists $0 \leq \sigma \leq 1$ such that $\mathcal{D}(\sqrt{n} \cdot v)$ is at most ε-far from the **Gaussian distribution** *$N(0, \sigma)$.*

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- the variance σ may be different for different eigenvectors
- $\sigma = 0$: localized eigenvectors if there exists any (this is open); the size of the support is $o(n)$, but the nonzero values are very large

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Approximate eigenvectors of random regular graphs

A vector v with $\|v\|_2 = 1$ is a **δ-approximate eigenvector** of A if for some $\lambda \in \mathbb{R}$ we have

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For every δ-approximate eigenvector v of G (with $\|v\|_2 = 1$ and entry sum 0) there exists $0 \leq \sigma \leq 1$ such that $\mathcal{D}(\sqrt{n} \cdot v)$ is at most ε-far from the **Gaussian distribution** *$N(0,\sigma)$.*

- $\sigma = 0$ is possible (by using eigenvectors of the infinite d-regular tree)
- another strengthening: the joint distribution at several vertices is also close to a Gaussian distribution.

Random regular graphs

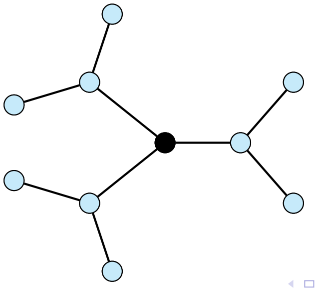
Fix $d \geq 3$.

$G = G(n,d)$: a uniformly chosen, simple d-regular graph on n vertices.

Local properties: $G(n,d)$ does not contain many small cycles with high probability – it looks like a tree.

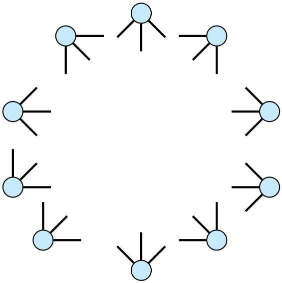
$G(n,d)$ tends to the **infinite d-regular tree T_d** in the Benjamini–Schramm (local) sense:

given n and r, the probability that the r-neighborhood of a uniformly chosen random vertex is a tree, tends to 1 as $n \to \infty$.



Configuration model

Bollobás (1984): assign d half-edges to each vertex, and choose a random perfect matching of the half-edges by connecting a uniform random pair at each step



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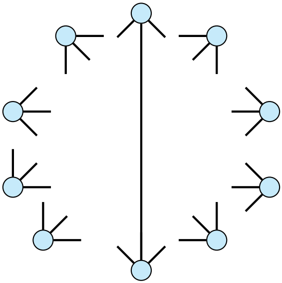

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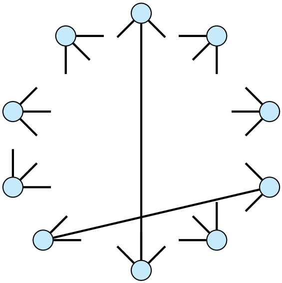

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Benjamini–Schramm convergence

(H_n): a sequence of finite graphs with all degrees at most Δ

$\mathcal{F}(\Delta, r)$: the set of connected rooted graphs with diameter at most $2r$

Definition (Benjamini–Schramm (local) convergence, 2001)

We say that (H_n) is convergent if for every r and $F \in \mathcal{F}(\Delta, r)$ the probability that the rooted r-neighborhood of a uniformly chosen vertex v of H_n is isomorphic to F is convergent as n tends to infinity.

Here the isomorphism has to be root-preserving.

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Examples (but: the limit is not necessarily a graph)

- a sequence of cycles or paths of length n tends to the infinite path;
- $n \times n$ grids tend to $\mathbb{Z}^2$;
- the sequence of random regular graphs $G(n, d)$ tend to the infinite d-regular tree.

Limits of colored regular graphs

S: finite set of colors

(H_n): a sequence of finite **d-regular graphs with colored vertices** (with the number of vertices tending to infinity but all degrees bounded by Δ)

$\mathcal{F}(\Delta, r, S)$: the set of connected rooted vertex-colored graphs with diameter at most $2r$

T_d: infinite d-regular tree with root o

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Invariant random process on T_d: to each vertex $v \in V(T_d)$, we assign a random variable X_v with values in S such that the joint distribution (X_v) is invariant under all automorphisms of the tree.

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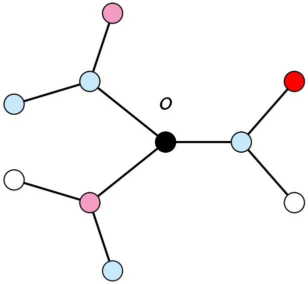
We say that (H_n) **converges locally** to $(X_v)_{v \in T_d}$ if for every r and $F \in \mathcal{F}(\Delta, r, S)$ the following holds. The probability that the colored rooted r-neighborhood of a uniformly chosen vertex v of H_n is isomorphic to F converges to the probability that the colored r-neighborhood of the root o of T_d is isomorphic to F.

Limits of colored regular graphs

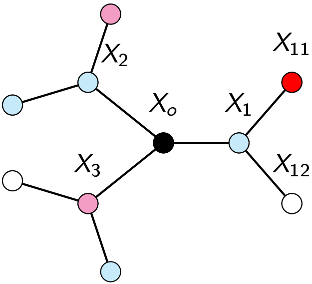
$r = 2$, F as below with the black vertex as the root:



The probability that the 2-neighborhood of a randomly chosen vertex is isomorphic to F should be convergent.

Limits of colored regular graphs

2-neighborhood of the root in an invariant random process:



$X_o, X_1, X_2, \ldots$ are random colors from S.

Typical processes

T_d: infinite d-regular tree, S: finite set

Definition (Typical process)

We say that an S-valued invariant random process $(X_v)_{v \in V(T_d)}$ is **typical** *if there exists a subsequence of the positive integers (n_k) with the following property.*

If, for each k independently, G_k is a random d-regular graph on n_k vertices, then, with probability 1, there exists a sequence of colorings $f_k : V(G_k) \to S$ such that (G_k, f_k) converges to $(X_v)_{v \in V(T_d)}$ locally as $k \to \infty$.

An $\mathbb{R}$-valued invariant random process is typical if it can be approximated by finite-valued typical processes in distribution.

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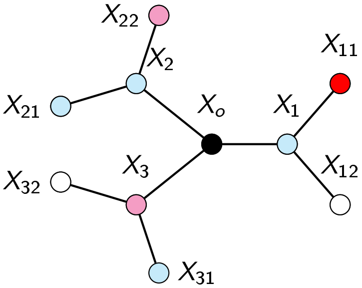
Open question: do we need subsequence in this definition?

Example for not typical process: alternating black and white with the color of the root chosen uniformly at random (Bollobás, 1984: a random d-regular graph is far from being bipartite with high probability, its independence ratio is smaller than $1/2$).

Entropy inequalities

Let $U \subset V(T_d)$ be a finite connected subgraph of the infinite tree. Then the entropy of the joint distribution $\underline{X} = (X_v)_{v \in U}$ will be denoted by $h(U)$:

$$h(U) = -\sum_F \mathbb{P}(\underline{X} = F) \cdot \log \mathbb{P}(\underline{X} = F).$$



Example: $h(B_2(o)) = h(X_o, X_1, X_2, X_{11}, X_{12}, \ldots, X_{32})$, where $B_2(o)$ is the 2-neighborhood of the root.

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Proposition

For every typical process the following hold:

(i)

$$\frac{d}{2} h(\bullet\!-\!\bullet) \geq (d-1) h(\bullet).$$

(ii)

$$h(B_1(\bullet)) \geq \frac{d}{2} h(\bullet\!-\!\bullet),$$

where $B_1(\bullet)$ is the 1-neighborhood of a vertex (a vertex and its d neighbors).

For factor of i.i.d. processes: Bowen (2008); the f-invariant is nonnegative; see also Rahman–Virág (2014).

Entropy inequalities

Proposition

For every typical process the following holds:

$$h(B_1(\bullet)) \geq \frac{d}{2} h(\overset{\bullet}{\underset{\bullet}{|}}),$$

where $B_1(\bullet)$ *is the* 1*-neighborhood of a vertex (a vertex and its* d *neighbors).*

Idea of the proof (similar to Bollobás's argument for the independence ratio):

- take the configuration model of the random regular graph;
- count the number of colorings that are close to the distribution of X_v on $B_1(\cdot)$;
- this is more than the total number of graphs.

Eigenvector processes

Definition

The family of random variables $(X_v)_{v \in T_d}$ *is an* **eigenvector process** *with eigenvalue* λ *if its distribution is invariant under* $\mathrm{Aut}(T_d)$*, and with probability* 1*,*

$$\lambda X_v = \sum_{w \sim v} X_w$$

holds for every $v \in V(T_d)$*.*

Harangi–Virág (2015): this exists for every $\lambda \in [-d, d]$.

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Theorem (B – Szegedy, 2016+)

If $(X_v)_{v \in V(T_d)}$ is a **typical eigenvector process** *with eigenvalue λ, it is nontrivial: $\mathbb{E}(X_o) = 0$ and it has finite second moments ($\mathrm{Var}(X_o) < \infty$),* **then it is a Gaussian process**.

Gaussian process: for every finite subset of $V(T_d)$, the joint distribution of the corresponding random variables is a multivariate Gaussian distribution.

Idea of the proof

Starting point: for a finite-valued typical process, we have

$$h(B_1(\bullet)) - \frac{d}{2} h(\vdots) \geq 0$$

Step 1: find entropy inequalities for larger balls ($B_r(F)$ denotes the set of vertices whose distance from a subset F is at most r), in particular, we have

$$h(B_{r+1}(\bullet)) - \frac{d}{2} h(B_r(\vdots)) \geq 0 \qquad (r \geq 1)$$

- for each vertex, write the state of all vertices in the r-neighborhood with some small extra randomization
- apply the $r = 1$ inequality to this new process, which is also typical

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Step 2: switch to smooth real-valued processes and differential entropy:

$$\mathbb{D}(X) = -\int f \log f,$$

where f is the density function of X

- we add an independent Gaussian eigenvector process to the original one of small variance – we get a typical eigenvector process with absolutely continuous marginals
- by a random discretization (using the grid $[-a, -a+1/a, -a+2/a, \ldots, a]$ for some large a), we obtain a finite-valued typical process that satisfies the entropy inequality
- by letting $a \to \infty$, we can prove that the inequality above holds for differential entropy

Idea of the proof

For a smooth typical process, we have

$$\mathbb{D}(B_{r+1}(\bullet)) - \frac{d}{2}\mathbb{D}(B_r(\bullet\!-\!\bullet)) \geq 0 \qquad (r \geq 1)$$

Step 3: show that for a *Gaussian* typical eigenvector process, we have

$$\lim_{r\to\infty} \mathbb{D}_{\mathrm{sp}}(B_{r+1}(\bullet)) - \frac{d}{2}\mathbb{D}_{\mathrm{sp}}(B_r(\bullet\!-\!\bullet)) = 0,$$

where $\mathbb{D}_{\mathrm{sp}}$ denotes the differential entropy calculated in the subspace on which the joint distribution is supported on.

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Step 4: show that for a smooth typical eigenvector process

$$\mathbb{D}_{\mathrm{sp}}(B_1(\bullet)) - \frac{d}{2}\mathbb{D}_{\mathrm{sp}}(\overset{\bullet}{\underset{\bullet}{|}})$$

maximizes this difference among all smooth distributions having finite variance and the appropriate covariance structure

Step 5: use heat equations to show that only the Gaussian distribution maximizes this difference (convolution by Gaussian noise increases entropy, but the original was itself maximal). It follows that the proces is Gaussian, and that a random lift of an eigenvector is close to some Gaussian eigenvector process.