Groups in graph theory I

David Conlon

28 August 2017

Expanders

Definition

A regular graph G is said to be an ϵ-expander if the expansion ratio

$$h(G) = \min_{\{S:|S|\leq n/2\}} \frac{e(S,S^c)}{|S|}$$

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Theorem (Pinsker)

Random regular graphs are expanders.

The adjacency matrix

The adjacency matrix A of a graph G on vertex set $\{1, 2, \ldots, n\}$ is the $n \times n$ matrix with entries given by

$$A_{uv} = \begin{cases} 0 & \text{if } uv \notin E(G); \\ 1 & \text{if } uv \in E(G). \end{cases}$$

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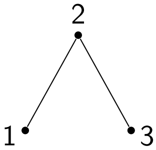

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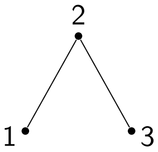

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Denote the eigenvalues of the adjacency matrix of A by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Expander Mixing

Expander Mixing Lemma

If G is a d-regular graph with n vertices for which all eigenvalues of the adjacency matrix, save the largest, have absolute value at most λ, then

$$|e(X,Y) - \frac{d}{n}|X||Y|| \leq \frac{\lambda}{n}\sqrt{|X||Y||X^c||Y^c|}$$

for all $X, Y \subseteq V(G)$.

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Corollary

Under the same hypotheses,

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$$\left|e(X,X^c) - \frac{d}{n}|X||X^c|\right| \le \frac{\lambda}{n}|X||X^c|$$

for all $X \subseteq V(G)$. Therefore, $\lambda \le d - \epsilon \implies$ expansion.

Proof of expander mixing

Let 1_X and 1_Y be the characteristic vectors of X and Y. Expand these in the orthonormal basis of eigenvectors of the adjacency matrix A, say $v_1, \dots, v_n$ with $Av_i = \lambda_i v_i$, writing

$$1_X = \sum_i \alpha_i v_i \text{ and } 1_Y = \sum_i \beta_i v_i.$$

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Then

$$e(X, Y) = 1_X^T A 1_Y = (\sum_i \alpha_i v_i)^T A (\sum_j \beta_j v_j) = \sum_i \lambda_i \alpha_i \beta_i.$$

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Noting that $v_1 = \mathbf{1}/\sqrt{n}$, we have $\alpha_1 = 1_X \cdot v_1 = |X|/\sqrt{n}$ and, similarly, $\beta_1 = |Y|/\sqrt{n}$. Therefore,

$$e(X, Y) = \frac{d}{n}|X||Y| + \sum_{i \geq 2} \lambda_i \alpha_i \beta_i.$$

Proof of expander mixing

To finish the proof, note, by the Cauchy–Schwarz inequality, that

$$|\sum_{i\geq 2}\alpha_i\beta_i\lambda_i| \leq \lambda|\sum_{i=2}^{n}\alpha_i\beta_i| \leq \lambda\sqrt{\sum_{i=2}^{n}\alpha_i^2\sum_{i=2}^{n}\beta_i^2}.$$

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By Parseval's identity,

$$\sum_{i\geq 2} \alpha_i^2 = \sum_{i} \alpha_i^2 - \alpha_1^2 = |X| - \frac{|X|^2}{n} = \frac{|X||X^c|}{n}$$

and, similarly, $\sum_{i\geq 2} \beta_i^2 = |Y||Y^c|/n$. The result follows.

Cayley graphs

Cayley graph

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Example

Paley graph - $G = \mathbb{Z}_p$, $S = \{x^2 : x \in \mathbb{Z}_p \setminus \{0\}\}$

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Let G be a finite group and let S be a random subset of size $100 \log |G|$. Then, with high probability, $\mathrm{Cay}(G, S)$ is an expander.

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Underlying idea

Let G be a finite group and let S be a random subset of size $d = 100 \log |G|$. Then, with high probability,

$$\lambda(Cay(G, S)) \leq d/2.$$

Alon–Roichman in the abelian case

Eigenvalues of an abelian group

If G is abelian, the eigenvalues of the Cayley graph $\text{Cay}(G, S)$ with $S = S^{-1}$ are

$$\sum_{s \in S} \chi(s)$$

where χ are the characters.

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Theorem (Alon–Roichman, 1994)

If G is a finite abelian group and S is a random subset of size d, then, with high probability, $\lambda(Cay(G, S)) = O(\sqrt{d \log n})$.

Proof of Alon–Roichman in the abelian case

The eigenvalue $\sum_{s \in S} \chi(s)$ is the sum of d random elements $\chi(s) + \chi(s^{-1})$ with expected value 0 and absolute value at most 2.

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$$\mathbb{P}[|\sum_{s \in S} \chi(s)| \geq t] \leq 4 \exp\{-ct^2/d\}.$$

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Taking $t = C\sqrt{d \log n}$ for C sufficiently large, we see that

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The result now follows by taking a union bound over all n characters χ.

Alon–Roichman in the general case

Proof idea similar, but use irreducible representations instead of characters.

For further details, see

Z. Landau and A. Russell, Random Cayley graphs are expanders, *Electron. J. Combin.* **11** (2004), R62.

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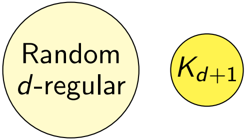
Observation

The converse of the expander mixing lemma is false.



Converses to expander mixing

Theorem (Bilu–Linial, 2006)

Suppose that G is a d-regular graph with n vertices such that

$$|e(X, Y) - \frac{d}{n}|X||Y|| \leq \eta\sqrt{|X||Y|}$$

for all $X, Y \subseteq V(G)$. Then $\lambda = O(\eta \log d)$.

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Theorem (Alon–Coja-Oghlan–Hàn–Kang–Rödl–Schacht, 2010)

Suppose that $(G_n)_{n\in\mathbb{N}}$ with $|G_n| = n$ is a sequence of graphs such that

$$|e(X,Y) - p|X||Y|| = o(pn^2)$$

for all $X, Y \subseteq V(G_n)$. Then one may remove a $o(1)$-fraction of the vertices to find a sequence of graphs $(G_n')_{n\in\mathbb{N}}$ with $\lambda = o(pn)$.

Converse to expander mixing for Cayley graphs

Theorem (Kohayakawa–Rödl–Schacht, 2017)

If G is an abelian group and $\mathrm{Cay}(G, S)$ satisfies

$$|e(X,Y) - \frac{d}{n}|X||Y|| \leq \epsilon dn$$

for all $X, Y \subseteq V(G)$, then all eigenvalues of $\mathrm{Cay}(G, S)$, save the largest, have absolute value $O(\epsilon d)$.

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Theorem (C.–Zhao, 2017)

If G is any group and Cay(G, S) satisfies

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One perspective on this result is to say that expansion in Cayley graphs bootstraps itself. Indeed, if

$$\left|e(X, Y) - \frac{d}{n}|X||Y|\right| \leq \epsilon dn$$

for all $X, Y \subseteq V(G)$, then $\lambda = O(\epsilon d)$, which implies that

$$\left|e(X, Y) - \frac{d}{n}|X||Y|\right| \leq \epsilon \frac{d}{n}\sqrt{|X||Y||X^c||Y^c|}.$$

Matrix norms

Given an $m \times n$ matrix A, define the spectral norm

$$\|A\| := \sup_{\substack{x \in \mathbb{R}^n \\ |x| \leq 1}} |Ax| = \sup_{\substack{x \in \mathbb{R}^m, y \in \mathbb{R}^n \\ |x|, |y| \leq 1}} |x^* A y|$$

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and the cut norm

$$\|A\|_{\text{cut}} := \sup_{S \subseteq [m], T \subseteq [n]} \Big| \sum_{s \in S, t \in T} a_{st} \Big|.$$

Relation with graphs

If G is an n-vertex d-regular graph with adjacency matrix A and J is the all-ones matrix, then

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Lesson

Suffices to understand the relationship between $\|\cdot\|$ and $\|\cdot\|_{\text{cut}}$.

Relaxing the cut norm

The cut norm is the same as its linear relaxation

$$\|A\|_{\text{cut}} = \sup_{x_1,\ldots,x_m,y_1,\ldots,y_n \in [0,1]} \Big| \sum_{s \in [m], t \in [n]} a_{st} x_s y_t \Big|.$$

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To see this, note that the expression inside the absolute value is linear in each of $x_1, \ldots, x_m, y_1, \ldots, y_n$ and hence its extremum is attained when all these variables are $\{0, 1\}$-valued.

Relaxing the cut norm

We further relax the cut norm by allowing each x_s and y_t to be numbers in $[-1, 1]$:

$$\|A\|_{\infty \to 1} := \sup_{x_1,\dots,x_m,y_1,\dots,y_n \in [-1,1]} \Big| \sum_{s \in [m], t \in [n]} a_{st} x_s y_t \Big|.$$

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This norm is equivalent to the cut norm:

$$\|A\|_{\text{cut}} \leq \|A\|_{\infty\to 1} \leq 4\|A\|_{\text{cut}}.$$

Indeed, write $x = x_+ - x_-$ and $y = y_+ - y_-$, where $x_+, x_- \in [0, 1]^m$ and $y_+, y_- \in [0, 1]^n$, and then apply the triangle inequality.

Relaxing the cut norm

Finally, we consider a semidefinite relaxation, which we shall refer to as the *Grothendieck norm*:

$$\|A\|_G := \sup_{x_1,\dots,x_m,y_1,\dots,y_n \in B(\mathbb{H})} \left| \sum_{s\in[m],t\in[n]} a_{st} \langle x_s, y_t \rangle \right|,$$

where $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is any Hilbert space and $B(\mathbb{H})$ is the unit ball in $\mathbb{H}$, containing all points in $\mathbb{H}$ of norm at most 1.

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A remarkable fact, proved by Grothendieck, and stated in this form by Lindenstrauss and Pełczyński, says that the Grothendieck norm is equivalent to the $\infty \to 1$ norm.

Grothendieck's inequality, 1953

There exists a constant $K_G < 1.78$ such that for all real-valued matrices A,

$$\|A\|_{\infty\to 1} \le \|A\|_G \le K_G \|A\|_{\infty\to 1}.$$

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Corollary

If A is the adjacency matrix of a Cayley graph,

$$\|A\|_{\text{cut}} \leq 4\|A\|_{\infty\to1} < 8\|A\|_G = 8n\|A\|.$$

Proof of key lemma, following Naor and Raghavendra

Let $f : G \to \mathbb{R}$. Choose $x, y : G \to \mathbb{R}$ with $\|x\|_2 \leq 1$ and $\|y\|_2 \leq 1$ such that

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$$\begin{aligned}
|G|^{-2}\|A\|_{\mathrm{G}} &\geq |\mathbb{E}_{g,h \in G} f(gh^{-1}) \langle x_g, y_h \rangle | \\
&= |\mathbb{E}_{g,h,a \in G} f(gh^{-1})x(ga)y(ha)| \\
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\end{aligned}$$

The opposite direction, $\|A\|_{\mathrm{G}} \leq |G|\|A\|$, follows from the Cauchy–Schwarz inequality.

Back to expanders

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Example (Lubotzky–Phillips–Sarnak, Margulis, 1988)

Let p and q be unequal primes, both congruent to 1 (mod 4), with p a quadratic residue modulo q. Let $PSL(2,q)$ be the projective special linear group of 2×2 matrices over the field of order q. For each vector $a = (a_0, a_1, a_2, a_3)$ such that a_0 is odd and positive, a_1, a_2 and a_3 are even and $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$, define $M_a \in PSL(2,q)$ by

$$M_a = \frac{1}{\sqrt{p}} \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix},$$

where i satisfies $i^2 \equiv -1 \pmod{q}$. Let $G_{p,q}$ be the Cayley graph with vertex set $PSL(2,q)$ defined by joining u and v if and only if $uv^{-1} = M_a$ for some a.

Ramanujan graphs

Theorem (Alon–Boppana, 1991)

For every n vertex d-regular graph

$$\lambda \geq 2\sqrt{d-1} - o(1).$$

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Solved for bipartite case by Marcus, Spielman and Srivastava.

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Also conjectured that the probability a random d-regular graph is Ramanujan lies strictly between 0 and 1.

Cayley graphs as expanders

Theorem (Breuillard–Green–Guralnick–Tao, 2014)

Let G be a finite simple group of Lie type (for example, $G = PSL_n(q)$) and let S be a random subset of size 2. Then $\text{Cay}(G, S)$ is an expander.

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Open problem

If $G = A_n$ and S is a random subset of fixed size, is $\mathrm{Cay}(G, S)$ an expander?

Thank you for listening!