Workshop on Graph and Hypergraph Domination

Problem Booklet

2017

Preliminary Schedule

Day 1 (5th June):
15:00- Arrival
18:30 - Dinner and other activities

Day 2 (6th June):
9:29 Waking up
8:30 - 9:30 Breakfast
10:00 - 10:45 Douglas F. Rall
11:00 - 11:45 Michael A. Henning
Lunch Break
14:00 - 14:45 Sandi Klavžar
15:00 - 15:45 Paul Dorbec
16:00 - 18:30 Talking about the problems (optional partitioning)
18:30 - Dinner and other activities

Day 3-5 (7th-9th June):
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Partitioning to Groups of 3-5 for the day
9:30 - 12:30 Work in Groups of 3-5
12:30 - 14:00 Lunch Break
14:00 Optional Repartitioning for the afternoon
14:00 - 17:00 Work in Groups of 3-5
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Last Day (10th June):
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Departure

List of Participants

Gábor Bacsó (MTA SZTAKI)

János Barát (U. of Pannónia, MTA-ELTE GAC)

Zoltán Blázsik (ELTE)

Boštjan Brešar (U. of Maribor, IMPM Ljubljana)

Csilla Bujtás (U. of Pannonia)

Dániel Gerbner (MTA Rényi)

Tanja Gologranc (U. of Maribor)

Marko Jakovac (U. of Maribor)

Balázs Keszegh (MTA Rényi)

Tim Kos (IMPM Ljubljana)

Gašper Košmrlj (IMPM Ljubljana)

Dániel Lenger (ELTE)

Tilen Marc (IMPM Ljubljana)

Abhishek Methuku (CEU)

Zoltán Lóránt Nagy (ELTE)

Dömötör Pálvölgyi (Cambridge)

Balázs Patkós (MTA Rényi)

Claire Pennarun (LaBRI, Bordeaux)

Casey Tompkins (MTA Rényi)

Zsolt Tuza (MTA Rényi, U. of Pannonia)

Máté Vizer (MTA Rényi)

Kirsti Wash (Trinity College, Hartford)

Game Total Domination

by Doug Rall

We assume throughout that all graphs under consideration have no isolated vertices. A vertex w in a graph *totally dominates* every vertex adjacent to w. That is, w totally dominates its open neighborhood, $N(w)$. The *total domination game* on a graph G is played by two players, Dominator and Staller, who alternate moves. A player's *turn* or *move* consists of choosing (or playing) a vertex u such that at least one vertex in $N(u)$ was not totally dominated by the set of vertices chosen previously in the game. Such a vertex is said to be *playable* or a *legal move*. Eventually the set of chosen vertices is a total dominating set of G, and the game ends. Dominator follows a strategy that will end the game in as few moves as possible while Staller employs a strategy to maximize the number of moves. The *game total domination number* $\gamma_{tg}(G)$ is the number of vertices chosen when Dominator starts the game (the *D-game*) and both players play optimally. The *Staller-start game total domination number*, $\gamma'_{tg}(G)$, is the number of vertices chosen when both players play optimally and Staller has the first move (the *S-game*).

The total domination game was introduced in [4] following much activity on the (ordinary) domination game. In the domination game a chosen vertex w dominates itself as well as $N(w)$, but the two players have the same goals in both games. The resulting graphical invariants are denoted by γ_g and γ'_g. As expected, there are similarities between the two games, but also some (perhaps) unusual differences. For example, for some graphs γ_g is much larger than γ_{tg} even though $\gamma(G) \leq \gamma_t(G)$ for every G. On the other hand, if G does not have a universal vertex, then $\gamma_{tg}(G) \leq 3\gamma_g(G) - 2$. (See [4].)

Problem 1. *Characterize the graphs G such that $\gamma_g(G) \leq \gamma_{tg}(G)$.*

In [4] it was shown that $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$ differ by at most 1. Henning and Rall [7] proved that $\gamma_{tg}(F) \leq \gamma'_{tg}(F)$ for any forest F with no components of order 1, but in general little is known about which pairs (k, ℓ) satisfying $|k - \ell| \leq 1$ can be $(\gamma_{tg}(G), \gamma'_{tg}(G))$ for some graph G. Since $\gamma_t(G) \leq \gamma_{tg}(G) \leq 2\gamma_t(G) - 1$ always holds, the following problems seem natural, if difficult in general.

Problem 2. *Characterize the graphs G such that $\gamma_t(G) = \gamma_{tg}(G)$.*

Problem 3. *Characterize those G such that $\gamma_{tg}(G) = 2\gamma_t(G) - 1$.*

A solution to Problem 2 when restricted to common graph classes would in itself be interesting. Recently in [7], Henning and Rall proved a characterization of those trees T for which $\gamma_t(T) = \gamma_{tg}(T)$. No characterization of the trees T for which $\gamma_{tg}(T) = 2\gamma_t(T) - 1$ is known.

Not surprisingly, the exact values of γ_{tg} and γ'_{tg} are not yet known for many classes of graphs. In fact, except for trivial classes such as complete multipartite graphs, the exact numbers are known only for cycles and paths. (See Dorbec and Henning [3].) Much of the research on the game total domination number is related to establishing upper bounds for $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$ in terms of the order of G. Since both vertices will be played in any component of order 2, we assume that all components of G have order at least 3. The first general upper bounds for the D-game and S-game total domination numbers were established by Henning, Klavžar and Rall [5]. They proved that if G is a graph of order n in which every component has more than two vertices, then $\gamma_{tg}(G) \leq \frac{4n}{5}$ and $\gamma'_{tg}(G) \leq \frac{4n+2}{5}$. The attack they used to prove these upper bounds was to modify a clever technique of Csilla Bujtás that she employed in her progress on the 3/5-Conjecture in the (ordinary) domination game. At each stage of playing the total domination game on G let D denote the set of vertices played so far. Relative to D each vertex v in G possesses one of several properties based on whether it belongs to D, whether it is totally dominated by D and whether its open neighborhood is totally dominated by D. If v and all its neighbors belong to $N(D)$, then v has no influence on the remainder of the game. In this case v can be deleted from the graph without affecting the future strategy of either player. Furthermore, if two playable vertices x and z are adjacent and are both totally dominated by D, then the edge xz also has no influence on the game going forward. Removing each such vertex v and each such edge xz after a legal move results in what is called the *residual* graph. Some subset A of vertices in the residual graph are totally dominated; to indicate this we denote the residual graph by G_A and say that the residual graph is a *partially totally dominated* graph. The general strategy to prove the $4n/5$ upper bound was to assign a weight of 4 to each vertex at the beginning of the game. When a vertex is played (i.e., added to D) the status of some vertices changes relative to the enlarged set of chosen vertices and the new, partially totally dominated residual graph. The weights of vertices are reduced based on their new status. It is then proved that Dominator has a strategy that guarantees the sum of all the vertex weights decreases by an average of at least 5 per move.

The authors of [5] posed the following conjecture, which has come to be known as the 3/4-Conjecture.

Conjecture 4. [**3/4-Conjecture**] *If G is a graph of order n in which every component has at least three vertices, then*

$$\gamma_{tg}(G) \le \frac{3n}{4} \quad \text{and} \quad \gamma'_{tg}(G) \le \frac{3n+1}{4}.$$

For positive integers r and s, let $G = rP_4 \cup sP_8$ and $H = G \cup P_5$. It can be shown that $\gamma_{tg}(G) = 3r + 6s = 3|V(G)|/4$ and that $\gamma'_{tg}(H) = 4 + 3r + 6s = (3|V(H)| + 1)/4$. This shows that the bounds in Conjecture 4 are tight infinitely often if in fact the conjecture is true.

Some progress has been made toward proving Conjecture 4. In [2] Bujtás, Henning and Tuza introduced a transversal game on hypergraphs. When specialized to a hypergraph whose edges are the open neighborhoods of a graph G with no isolated vertices, this game yields $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$. As a result they were able to verify the 3/4-Conjecture for the class of graphs with minimum degree at least 2. Using yet another variation of the "weighting argument," Henning and Rall [6] proved that if G is a graph such that the sum of the degrees of every pair of adjacent vertices is at least 4 and no pair of leaves in G are at a distance exactly 4 apart, then the inequalities in Conjecture 4 hold for G. In fact, they showed that if G has minimum degree at least 2, then Dominator can follow a greedy strategy (i.e., always choose a vertex that gives a maximum reduction of weight with respect to their vertex weight assignments) and end the game in at most $3n/4$ moves. An obvious problem generalizing this to larger minimum degrees is the following.

Problem 5. *If a graph G of order n has minimum degree at least $\delta \ge 2$, prove sharp upper bounds on $\gamma_{tg}(G)$ and $\gamma'_{tg}(G)$ in terms of n and δ.*

The following is a list of topics that, to my knowledge, have not been studied for the total domination game but the corresponding topic has been investigated for the ordinary domination game. Consequently, many open questions and problems arise as a result of this list.

1. The behavior of γ_{tg} and γ'_{tg} on disjoint unions of graphs.

2. The effect of edge removal or vertex removal on γ_{tg} and γ'_{tg}.

3. Game total domination on spanning subgraphs and spanning trees.

4. The complexity of computing γ_{tg} and γ'_{tg}.

5. A characterization of graphs with small game total domination number.

If the vertices in the total domination game are chosen without regard to a strategy but only to insure that each vertex played enlarges the open neighborhood of the set of chosen vertices, then the resulting sequence is called a *total dominating sequence*. The length of the longest such sequence in a graph G is called the *Grundy total domination number* of G; it is denoted by $\gamma_{\rm gr}^t(G)$. Brešar, Henning and Rall introduced total dominating sequences in [1]. They characterized the graphs G for which $\gamma_{\rm gr}^t(G) = |V(G)|$ and those for which $\gamma_{\rm gr}^t(G) = 2$. In the case of trees T of order n, $\gamma_{\rm gr}^t(T) = n$ if and only if T has a perfect matching. In addition they showed that if T is a nontrivial tree of order n with no strong support vertex, then $\gamma_{\rm gr}^t(T) \geq \frac{2}{3}(n+1)$, and they characterized the trees that achieve equality. If $k \geq 3$ and G is a connected, k-regular graph of order n other than $K_{k,k}$, then they proved $\gamma_{\rm gr}^t(G) \geq (n + \lceil \frac{k}{2} \rceil - 2)/(k-1)$ if G is not bipartite and $\gamma_{\rm gr}^t(G) \geq (n + 2\lceil \frac{k}{2} \rceil - 4)/(k-1)$ if G is bipartite. For $k = 3$ (resp. $k = 4$) and G not $K_{k,k}$, the above bounds are $\frac{1}{2}n$ and $\frac{1}{3}n$ respectively, in both the bipartite and non-bipartite cases. Examples are given in [1] to show the bounds are tight in these cases.

Problem 6. *Characterize the connected* 3*-regular graphs* G *of order* n *such that* $\gamma_{\rm gr}^t(G) = \frac{1}{2}n$*.*

Problem 7. *Characterize the connected* 4*-regular graphs* G *of order* n *such that* $\gamma_{\rm gr}^t(G) = \frac{1}{3}n$*.*

It is straightforward to show that $\gamma_{\rm gr}^t(G) \leq n - \delta(G) + 1$ for any graph G of order n and minimum degree $\delta(G) \geq 1$.

Problem 8. *Characterize the graphs* G *of order* n *such that* $\gamma_{\rm gr}^t(G) = n - \delta(G) + 1$ *for* $\delta(G) \geq 1$*.*

In [1] it is shown that complete multipartite graphs G are the only graphs with $\gamma_{\rm gr}^t(G) = 2 = \gamma_t(G)$ and that no graph with total domination number 3 also has Grundy total domination number 3. However, it is also shown that there are infinitely many connected graphs G with $\gamma_{\rm gr}^t(G) = 4 = \gamma_t(G)$.

Problem 9. *For each* $k \geq 4$*, characterize the connected graphs* G *such that* $\gamma_{\rm gr}^t(G) = \gamma_t(G) = k$*.*

References

[1] Boštjan Brešar, Michael A. Henning and Douglas F. Rall. Total dominating sequences in graphs. Discrete Math. 339 (2016), 1665–1676.

[2] Csilla Bujtás, Michael A. Henning and Zsolt Tuza. Transversal game on hypergraphs and the 3/4-conjecture on the total domination game. SIAM J. Discrete Math. 30 (2016), no. 3, 1830–1847.

[3] Paul Dorbec and Michael A. Henning. Game total domination for cycles and paths. Discrete Appl. Math. 208 (2016), 7–18.

[4] Michael A. Henning, Sandi Klavžar and Douglas F. Rall. Total version of the domination game. Graphs Combin. 31 (2015), no. 5, 1453–1462.

[5] Michael A. Henning, Sandi Klavžar and Douglas F. Rall. The 4/5 upper bound on the game total domination number. Combinatorica, to appear, 1–29.

[6] Michael A. Henning and Douglas F. Rall. Progress towards the total domination game 3/4-conjecture. Discrete Math. 339 (2016), no. 11, 2620–2627.

[7] Michael A. Henning and Douglas F. Rall. Trees with equal total domination and game total domination numbers. submitted (2017), 1–23.

On the Fractional Total Domatic Number of a Graph

by Michael A. Henning

1 The Fractional Total Domatic Number

We consider here the fractional analogue of the total domatic number of a graph. A *total dominating set* of a graph G with no isolates is a set S of vertices such that every vertex in G is adjacent to a vertex in S. The *total domination number*, $\gamma_t(G)$, of G is the smallest cardinality of a total dominating set. The *total domatic number*, $tdom(G)$, of G is the maximum number of disjoint total dominating sets [6]. This can also be considered as a coloring of the vertices such that every vertex has a neighbor of every color (and has been called the *coupon coloring problem* [5]). Zelinka [15] showed that there are graphs with arbitrarily large minimum degree without two disjoint total dominating sets. Heggernes and Telle [8] showed that the decision problem to decide for a given graph G if $tdom(G) \geq 2$ is NP-complete, even for bipartite graphs. In contrast, several researchers, such as Aram et al. [1], studied the total domatic number of a k-regular graph; in particular, Chen et al. [5] showed that such graphs have total domatic number at least $(1 - o(1))k/\ln k$.

Goddard and Henning [9] define a *total dominating family* $\mathcal{F}$ of a graph G as a family of (not necessarily distinct) total dominating sets of G. We denote by $r_{\mathcal{F}}$ the maximum times any vertex of G appears in $\mathcal{F}$, and define the *effective ratio* of the family $\mathcal{F}$ as the ratio of the number of sets in $\mathcal{F}$ to $r_{\mathcal{F}}$. The *fractional total domatic number* $FTD(G)$ is then defined as the supremum of the effective ratio taken over all total dominating families. That is,

$$FTD(G) = \sup_{\mathcal{F}} \frac{|\mathcal{F}|}{r_{\mathcal{F}}}.$$

Like other fractional parameters, one can show that the supremum can be achieved. The following result was first observed in [9].

Theorem 1 (([9]))**.** *If a graph G of order n has minimum degree $\delta \geq 1$, then the following hold.*

(a) $tdom(G) \leq FTD(G) \leq \dfrac{n}{\gamma_t(G)}$.

(b) $\dfrac{n}{n - \delta + 1} \leq FTD(G) \leq \delta$.

As an immediate consequence of Theorem 1(b), if a graph G has minimum degree $\delta \geq 2$, then $FTD(G) > 1$. However, as shown in [9], there are graphs G with arbitrarily large minimum degree with $FTD(G) < 1 + \epsilon$. We consider next some specific families of graphs.

1.1 Claw-Free Graphs

It is shown in [9] that if G is a claw-free graph with $\delta \geq 2$, then $FTD(G) \geq \frac{3}{2}$. This lower bound is somewhat best possible: the graphs K_3 and C_6 have fractional total domatic number exactly 3/2. However, we believe that the lower bound should be improvable asymptotically.

Question 1. ([9]) *Is it true that if G is a connected, claw-free graph with $\delta \geq 2$, then $FTD(G) \geq 2 - o(1)$ and/or there is a partition (T_1, T_2) of the vertex set such that every vertex except possibly two has a neighbor in both T_1 and T_2?*

Question 2. ([9]) *Is it true that if G is a claw-free graph with $\delta \geq 3$, then $tdom(G) \geq 2$?*

1.2 Triangulated discs

Recall that a *triangulated disc* is a (simple) planar graph all of whose faces are triangles, except possibly for the outer face. As a consequence of results in [9, 12], we have that if G is a triangulated disc, then $FTD(G) \geq \frac{3}{2}$. This lower bound is tight, in that there exist triangulated discs G of arbitrarily large order satisfying $FTD(G) = \frac{3}{2}$.

We next consider (simple) planar triangulations or equivalently maximal planar graphs (that is, triangulated discs where the outer face is a triangle). Since every planar triangulation is a triangulated disc, our earlier result implies that every planar triangulation G satisfies $FTD(G) \geq \frac{3}{2}$. We believe this lower bound can be improved significantly.

Conjecture 3. ([9]) *If G is a planar triangulation of order at least 4, then $tdom(G) \geq 2$.*

Conjecture 3 has been established for a few cases.

Theorem 2. ([9]) *If G is a planar triangulation, then the following hold.*
(a) *If every vertex of G has odd degree, then $tdom(G) \geq 2$.*
(b) *If the dual of G is hamiltonian, then $tdom(G) \geq 2$.*

A computer search suggests the following conjecture.

Conjecture 4. *Every planar triangulation with at least four vertices has a proper 4-coloring (C_1, C_2, C_3, C_4) such that $C_1 \cup C_2$ and $C_3 \cup C_4$ are total dominating sets.*

If one imposes larger minimum degree, it appears even more can be said.

Conjecture 5. ([9]) *If G is a planar triangulation with $\delta(G) \geq 4$, then $tdom(G) \geq 3$.*

Perhaps it is true that every triangular disc with minimum degree at least 3 has two disjoint total dominating sets. It is not true that every triangular disc with minimum degree at least 4 has three disjoint total dominating sets: the icosahedron minus a vertex is an example, and there is an example of order 10. But maybe there are only finitely many exceptions.

2 The Fractional Disjoint Transversal Number

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover*, *hitting set* or *blocking set*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. See for example [2, 3, 4]. A hypergraph H is 2-*colorable* if there is a 2-coloring of the vertices such that each hyperedge contains two vertices of distinct colors; that is, there is no monochromatic hyperedge. So, the question of when a hypergraph has two disjoint transversals is the same as whether the hypergraph has a 2-coloring. More generally, Kostochka and Woodall [11] defined a panchromatic k-coloring of a hypergraph as a coloring with k colors such that every hyperedge contains each color. This is equivalent to a partition into k disjoint transversals. We denote by $disj_\tau(H)$ the *disjoint transversal number* of a hypergraph H, which is the maximum number of disjoint transversals in H.

Analogous to the fractional total domatic number, one can define the fractional disjoint transversal number. A *transversal family* $\mathcal{F}$ of a hypergraph H is a family of transversals of H. Given a hypergraph H and a transversal family $\mathcal{F}$, we define the *effective transversal-ratio* of the family $\mathcal{F}$ as the ratio of the number of sets in $\mathcal{F}$ over the maximum times $r_{\mathcal{F}}$ any element appears in $\mathcal{F}$. The *fractional disjoint transversal number* $FDT(H)$ is the supremum of the effective transversal-ratio taken over all transversal families. That is,

$$FDT(H) = \sup_{\mathcal{F}} \frac{|\mathcal{F}|}{r_{\mathcal{F}}}.$$

The following results were first observed in [9].

Observation 1. ([9]) *For every isolate-free hypergraph H of order n,*

$$disj_{\tau}(H) \le FDT(H) \le \frac{n}{\tau(H)}.$$

Associated with a graph G, one can define the *open neighborhood hypergraph* of G as the hypergraph whose vertex set is $V(G)$ and whose hyperedges are the open neighborhoods of vertices in G. The fractional total domatic number of an isolate-free graph is precisely the fractional disjoint transversal number of its open neighborhood hypergraph, as observed in [9].

Observation 2. ([9]) *If G is an isolate-free graph, then $FTD(G) = FDT(H)$, where H is the open neighborhood hypergraph of G.*

Using a connection with not-all-equal 3-SAT, the following result is proven in [10].

Theorem 3. ([10]) *If H is a 3-regular 3-uniform hypergraph of order n, then there exists $2k$ transversals in H such that any vertex in H belongs to at most k of them for some $k \ge 1$.*

As an immediate consequence of Theorem 3, if H is a 3-regular 3-uniform hypergraph, then $FDT(H) \ge 2$. We remark that this bound is tight. By the connection with the ONH of a graph G (see Observation 2), we have the following result.

Theorem 4. ([10]) *If G is a connected cubic graph, then $FTD(G) \ge 2$.*

Theorem 4 is in contrast to the fact that not every cubic graph has two disjoint total dominating sets. For example, it is known that the Heawood graph is the smallest such example. (For more information see for example McCuaig [13], or Gropp [7].) In contrast, Thomassen [14] showed that, for $r \ge 4$, every r-regular graph has two disjoint total dominating sets. The following general lower bound on the fractional disjoint transversal number of a k-regular k-uniform hypergraph for all $k \ge 3$ is given in [10].

Theorem 5. ([10]) *For all $k \ge 3$, if H is a k-regular k-uniform hypergraph, then*

$$FDT(H) \ge \frac{1}{1 - \left(\frac{k-1}{k}\right)\left(\frac{1}{k}\right)^{\frac{1}{k-1}}}.$$

In the special case when $k = 5$, this implies that $FDT(H) > 2.1505$. When $k = 4$, the hypergraph H is 2-colorable, and so, by Observation 1, $FDT(H) \ge disj_{\tau}(H) \ge 2$. It would be interesting to establish whether strict inequality holds in this case when $k = 4$.

Question 6. ([9]) *Is it true that if H is a 4-regular 4-uniform hypergraph, then $FDT(H) > 2$?*

We suspect that the following stronger conjecture holds.

Conjecture 7. ([10]) *If H is a 4-regular 4-uniform hypergraph, then $FDT(H) \ge \frac{7}{3}$, with equality if and only if H is the complement of the Fano plane.*

References

[1] H. Aram, S.M. Sheikholeslami and L. Volkmann, *On the total domatic number of regular graphs*, Trans. Comb. **1** (2012), 45–51.

[2] C. Berge, *Packing problems and hypergraph theory: a survey*, Ann. Discrete Math. **4** (1979), 3–37.

[3] C. Bujtás, M.A. Henning, and Zs. Tuza, *Transversals and domination in uniform hypergraphs*, European J. Combin. **33** (2012), 62–71.

[4] C. Bujtás, M.A. Henning, Zs. Tuza, and A. Yeo, *Total transversals and total domination in uniform hypergraphs*, Electronic J. Combin. **21(2)** (2014), #P2.24.

[5] B. Chen, J.H. Kim, M. Tait, and J. Verstraete, *On coupon colorings of graphs*, Discrete Applied Mathematics **193** (2015), 94–101.

[6] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), 211–219.

[7] H. Gropp, *Blocking set free configurations and their relations to digraphs and hypergraphs*, Discrete Math. **165/166** (1997), 359–370.

[8] P. Heggernes and J. A. Telle, *Partitioning graphs into generalized dominating sets*, Nordic J. Comput. **5** (1998), 128–142.

[9] W. Goddard and M. A. Henning, *Thoroughly distributed colorings*, manuscript.

[10] M.A. Henning and A. Yeo, Fractional disjoint transversals in hypergraphs, manuscript.

[11] A.V. Kostochka and D.R. Woodall, *Density conditions for panchromatic colourings of hypergraphs*, Combinatorica **21** (2001), 515–541.

[12] L.R. Matheson and R.E. Tarjan, *Dominating sets in planar graphs*, European J. Combin. **17** (1996), 565–568.

[13] W. McCuaig, *Even dicycles*, J. Graph Theory **35** (2000), 46–68.

[14] C. Thomassen, *The even cycle problem for directed graphs*, J. Amer. Math. Soc. **5** (1992), 217–229.

[15] B. Zelinka, *Total domatic number and degrees of vertices of a graph*, Math. Slovaca **39** (1989), 7–11.

Selected Problems on Domination Game

by Sandi Klavžar

3 The game and the game domination number

A vertex u in a graph G *dominates* a vertex v if $u = v$ or u is adjacent to v. A *dominating set* of G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The size of a smallest dominating set of G is the *domination number* $\gamma(G)$ of G.

The domination game was introduced in 2010 as follows [6]. The game is played on a graph G by two players named *Dominator* and *Staller*. They take turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated until no move is possible. The score of the game is the total number of vertices chosen by them in this game. The players have opposite goals: Dominator wants to minimize the score and Staller wants to maximize it. A game is called a *D-game* (resp. *S-game*) if Dominator (resp. Staller) has the first move. The *game domination number* $\gamma_g(G)$ of G is the score of a D-game played on G assuming that both players play optimally, the *Staller-start game domination number* $\gamma_g'(G)$ is the score of an optimal S-game.

4 The $3/5$-conjecture

If $G = (V(G), E(G))$ is a graph, we will use $n(G)$ to denote the order of G, that is, $n(G) = |V(G)|$. Kinnersley, West, and Zamani [15, Conjecture 6.2] posed the conjecture that $\gamma_g(G) \leq 3n(G)/5$ holds for any isolate-free graph G. (Related conjectures were stated also for the S-game, as well as for both games played on forests.) This conjecture is now known as the $3/5$-*conjecture*. Bujtás [8, 9] developed an innovative discharging-like method to attack this conjecture. Using the method, the conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [12]. Along these lines Schmidt [22] determined a largest known class of trees for which the conjecture holds. Moreover, Marcus and Peleg reported in arXiv [21] that the conjecture holds on all isolate-free forests. The following case is thus still open.

Problem 1. *Prove (or disprove) the* $3/5$*-conjecture for the class of graphs containing pendant vertices.*

The proof of Marcus and Peleg [21] is quite technical, hence assuming that the reviewers will confirm its correctness, a simpler proof of the 3/5-conjecture for trees would still be of interest!

An interesting related problem is which graphs attain the 3/5-bound. A significant progress on this problem has been made on the class of trees. Using a computer all such trees up to 20 vertices were found [4]. For instance, on 20 vertices there are (only) ten trees that attain the 3/5-bound. A construction that yields an infinite family of trees that attain the 3/5-bound was also given. Henning and Löwenstein [13] followed with the following wider construction. Call a tree T to be a 2-*wing* if T

(i) has maximum degree at most 4,

(ii) has no vertex of degree 3, and

(iii) the vertices of degree 2 in T are precisely the support vertices of T, except for one vertex of degree 2 in T.

This exceptional vertex of degree 2 in T (that is, the vertex that is not a support vertex) is called the *gluing vertex* of T. (Note that the smallest 2-wing is P_5, its central vertex being the gluing vertex.) Now, a tree T belongs to the family $\mathcal{T}$ if T is obtained from $k \geq 1$ vertex-disjoint 2-wings by adding $k-1$ edges between the gluing vertices. Henning and Löwenstein proved that every tree $T \in \mathcal{T}$ attains the 3/5-bound and posed:

Conjecture 2. [13, Conjecture 1] *If F is an isolate-free forest on n vertices satisfying $\gamma_g(F) = 3n/5$, then every component of F belongs to the family $\mathcal{T}$.*

In this respect the following question due to Cs. Bujtás (personal communication) is also relevant:

Question 3. *Do there exist 2-connected graphs G different from C_5 for which $\gamma_g(G) = 3n(G)/5$ holds?*

5 Edge- and vertex-removal

In [1] it was proved that if $e \in E(G)$, then $|\gamma_g(G) - \gamma_g(G-e)| \leq 2$ and $|\gamma_g'(G) - \gamma_g'(G-e)| \leq 2$, and that each of the possibilities is realizable by connected graphs G for all values of $\gamma_g(G)$ and $\gamma_g'(G)$ larger than 5. For the remaining small values it was either proved that realizations are not possible or realizing examples provided. It was also proved that if $v \in V(G)$, then $\gamma_g(G) - \gamma_g(G-v) \leq 2$ and $\gamma_g'(G) - \gamma_g'(G-v) \leq 2$. Possibilities are again realizable by connected graphs in almost all the cases.

The following problem was posed in [1, Problem 4.1]: given a positive integer k, can one find a general upper and lower bound for $\gamma_g(G) - \gamma_g(G_k)$, where G_k is obtained from a graph G by deletion of k edges from G? The problem was solved by Henning and Kinnersley [12, Theorem 3.4] as follows: If G and H are graphs on a common vertex set V, then

$$|\gamma_g(G) - \gamma_g(H)| \leq k + \epsilon\,, \tag{5.1}$$

where k is the size of the symmetric difference of $E(G)$ and $E(H)$, and $\epsilon = 0$ if k is even and 1 if k is odd. Another problem posed in [1] is:

Problem 4. [1, Problem 4.2] *Which of the subsets of $\{-2,-1,0,1,2\}$ can be realized as*

$$\{\gamma_g(G) - \gamma_g(G-e):\ e \in E(G)\}$$

within the family of all (respectively connected) graphs G?

A partial answer to the latter problem follows from (5.1): since for all edges $e, e' \in E(G)$ we have $|\gamma_g(G-e) - \gamma_g(G-e')| \leq 2$, each such set must be contained in $\{-2,-1,0\}$, $\{-2,0,1\}$, or $\{0,1,2\}$.

6 Domination game critical graphs

A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is, they need not be dominated in the rest of the game. For $S \subseteq V(G)$ of a graph G the partially dominated graph in which vertices from S are already dominated is denoted $G|S$. If $S = \{v\}$, then the notation can be simplified to $G|v$.

A graph G is *domination game critical*, shortly γ_g*-critical*, if $\gamma_g(G) > \gamma_g(G|v)$ holds for every $v \in V(G)$. This concept was introduced in [10] where among other results γ_g-critical graphs with $\gamma_g = 2$ and with $\gamma_g = 3$ are characterized, and for each n the (infinite) class of all γ_g-critical ones among the nth powers C_N^n of cycles are determined.

A concept complementary to the domination game criticality is the one from the following problem.

Problem 5. [10, Problem 13] *Study the graphs G for which $\gamma_g(G) = \gamma_g(G|v)$ holds for every $v \in V(G)$. In particular, establish their connections with the γ_g-critical graphs.*

7 Computational aspects

Computational aspects of the domination game were studied in [3, 17]. In [17] it was shown that for a given integer m and a given graph G, deciding whether $\gamma_g(G) \leq m$ can be done in $\mathcal{O}(\Delta(G) \cdot n(G)^m)$ time. This result was complemented in [3] by proving that the complexity of verifying whether the game domination number of a graph is bounded by a given integer is in the class of PSPACE-complete problems, implying that every problem solvable in polynomial space (possibly with exponential time) can be reduced to this problem. In particular, this shows that the game domination number of a graph is harder to compute than any other classical domination parameter (which are generally NP-hard), unless NP=PSPACE.

On the (possible) positive side, the following problem was posed:

Question 6. [3, Question 1] *Can the game domination number of (proper) interval graphs be computed in polynomial time?*

This question can be made more general as follows.

Question 7. [3, Question 1] *For which non-trivial families of graphs can the game domination number be computed in polynomial time?*

Here a "non-trivial family" must be taken with a grain of salt. For instance, the family of graphs which contain universal vertices would certainly not be such, because for each such graph G we have $\gamma_g(G) = 1$. So "non-trivial family" would be a family on which the computation/determination of the game domination number is "non-trivial". For instance, paths and cycles, caterpillars, and powers of cycles, classify as "non-trivial". That this is indeed the case see [14, 19] for paths and cycles, [18] for caterpillars, and [10] for powers of cycles.

8 Graph with trivial game domination numbers

If G is a graph, then clearly $\gamma_g(G) \geq \gamma(G)$ holds. Nadjafi-Arani, Siggers, and Soltani [20] called a graph G to be *D-trivial* if $\gamma_g(G) = \gamma(G)$, and *S-trivial* if $\gamma_g'(G) = \gamma(G)$. In [20] D-trivial forests (as well as S-trivial forests) are characterized and the following conjecture posed:

Conjecture 8. [20, Conjecture 6.1] *Any connected D-trivial graph is either a tree or has girth at most* 7.

9 Bluffing in the domination game

Call a graph to be a *bluff graph* if every vertex is an optimal start vertex in D-Game as well as in S-Game. If $\gamma_g'(G) = \gamma(G) - 1$, then G is called a MINUS *graph*. (C_6 is a sporadic example of a MINUS graph.) In [2] it is proved that every MINUS graph is a bluff graph and a non-trivial infinite family of MINUS graphs is established. In addition, several generalized Petersen graphs that are bluff graphs but not vertex-transitive are determined.

Call a graph to be a *double bluff graph*, if it is a bluff graph, where after the first move any legal answer is an optimal second move for any player. In other words, the first two moves are arbitrary, provided they are legal. In [2] it is proved that Kneser graphs $K(n, 2)$, $n \geq 6$, are double bluff and that Hamming graphs are not double bluff. So here is a natural related problem.

Problem 9. *Characterize double bluff graphs. If this is too difficult, then find additional families of double bluff graphs.*

10 No-MINUS graphs

In [11] the following concept was introduced: a graph G is a *no-*MINUS *graph* if for any $S \subseteq V(G)$ we have $\gamma_g(G|S) \leq \gamma_g'(G|S)$. The intuition behind this definition is that no player should get any advantage by passing in a no-MINUS graph. In [15] it was proved that forests are no-MINUS graphs, while in [11] this was proved for two additional families of graphs including dually chordal graphs.

Problem 10. *Characterize no-*MINUS *graphs. If this is too difficult, then find additional families of no-*MINUS *graphs.*

11 Further reading

For additional aspects of the domination game we refer to:

- [7], for the behaviour of the game played on trees and spanning subgraphs;
- [5], for the domination game played on the so-called guarded subgraphs;
- [11], where the game is investigated on unions of graphs;
- [16], where graphs with small game domination numbers are described; and
- [18], where different realizations of the game domination number are provided.

References

[1] B. Brešar, P. Dorbec, S. Klavžar, G. Košmrlj, Domination gama: effect of edge- and vertex-removal, Discrete Math. 330 (2014) 1–10.

[2] B. Brešar, P. Dorbec, S. Klavžar, G. Košmrlj, How long can one bluff in the domination game? Discuss. Math. Graph Theory 37 (2017) 337–352.

[3] B. Brešar, P. Dorbec, S. Klavžar, G. Košmrlj, G. Renault, Complexity of the game domination problem, Theoret. Comput. Sci. 648 (2016) 1–7.

[4] B. Brešar, S. Klavžar, G. Košmrlj, D. F. Rall, Domination game: extremal families of graphs for 3/5-conjectures, Discrete Appl. Math. 161 (2013) 1308–1316.

[5] B. Brešar, S. Klavžar, G. Košmrlj, D. F. Rall, Guarded subgraphs and the domination game, Discrete Math. Theor. Comput. Sci. 17 (2015) 161–168.

[6] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.

[7] B. Brešar, S. Klavžar, D. F. Rall, Domination game played on trees and spanning subgraphs, Discrete Math. 313 (2013) 915–923.

[8] Cs. Bujtás, Domination game on forests, Discrete Math. 338 (2015) 2220–2228.

[9] Cs. Bujtás, On the game domination number of graphs with given minimum degree, Electron. J. Combin. 22 (2015) #P3.29.

[10] Cs. Bujtás, S. Klavžar, G. Košmrlj, Domination game critical graphs, Discuss. Math. Graph Theory 35 (2015) 781–796.

[11] P. Dorbec, G. Košmrlj, and G. Renault, The domination game played on unions of graphs, Discrete Math. 338 (2015) 71–79.

[12] M. A. Henning, W. B. Kinnersley, Domination game: A proof of the 3/5-conjecture for graphs with minimum degree at least two, SIAM J. Discrete Math. 30 (2016) 20–35.

[13] M. A. Henning, Ch. Löwenstein, Domination game: extremal families for 3/5-conjecture for forests, Discuss. Math. Graph Theory 37 (2017) 369–381.

[14] W. B. Kinnersley, D. B. West, R. Zamani, Game domination for grid-like graphs, manuscript, 2012.

[15] W. B. Kinnersley, D. B. West, R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (2013) 2090–2107.

[16] S. Klavžar, G. Košmrlj, S. Schmidt, On graphs with small game domination number, Appl. Anal. Discrete Math. 10 (2016) 30–45.

[17] S. Klavžar, G. Košmrlj, S. Schmidt, On the computational complexity of the domination game, Iran. J. Math. Sci. Inform. 10 (2015) 115–122.

[18] G. Košmrlj, Realizations of the game domination number, J. Comb. Optim. 28 (2014) 447–461.

[19] G. Košmrlj, Domination game on paths and cycles, Ars Math. Contemp. 13 (2017) 125–136.

[20] M. J. Nadjafi-Arani, M. Siggers, H. Soltani, Characterisation of forests with trivial game domination numbers, J. Comb. Optim. 32 (2016) 800–811.

[21] N. Marcus, D. Peleg, The domination game: proving the 3/5 conjecture on isolate-free forests, arXiv:1603.01181 [cs.DS].

[22] S. Schmidt, The 3/5-conjecture for weakly $S(K_{1,3})$-free forests, Discrete Math. 339 (2016) 2767–2774.

Power Domination

by Paul Dorbec

Power domination was described as a graph theoretical problem by Haynes et al. in [12]. The problem is motivated by the requirement for constant monitoring of power systems by placing a minimum number of phasor measurement units (PMU) in the network. A PMU placed at a bus measures the voltage of the bus plus the current phasors at that bus. Using Ohm and Kirchhoff current laws, it is then possible to infer from initial knowledge of the status of some part of the network the status of new branches or buses. This can be simply described as a graph optimisation problem similar to domination with addition of a propagation behaviour.

We here describe directly the generalized graph parameter named k-power domination, as defined in [6], following the detailed definition of propagation from [1]. We use notations $N(u)$ and $N[u]$ to denote respectively the open neighbourhood and closed neighbourhood of a vertex u, defined by $N(u) = \{v \mid uv \in E\}$ and $N[u] = N(u) \cup \{u\}$. By extension, for a subset S of vertices, $N[S] = \bigcup_{u \in S} N[u]$.

Definition 1. Let G be a graph, $S \subseteq V(G)$ and k a non-negative integer. We define the sets $\left(\mathcal{P}_k^i(S)\right)_{i \geq 0}$ of vertices monitored by S at step i by the following rules.

- $\mathcal{P}_k^0(S) = N[S]$.
- $\mathcal{P}_k^{i+1}(S) = \bigcup N[v]$, $v \in \mathcal{P}_k^i(S)$ such that $\left|N[v] \setminus \mathcal{P}_k^i(S)\right| \leq k$.

Necessarily from this definition, for any $i \geq 0$, $\mathcal{P}_k^i(S) \subseteq \mathcal{P}_k^{i+1}(S)$. Indeed, there exists some set S' (equal to S when i is 0) such that $\mathcal{P}_k^i(S) = N[S']$. Any vertex v in S' satisfies that $\left|N[v] \setminus \mathcal{P}_k^i(S)\right| = 0 \leq k$, and thus $N[S'] \subseteq \mathcal{P}_k^{i+1}(S)$. Since $\mathcal{P}_k^i \subseteq V(G)$, the sequence $(\mathcal{P}_k^i)_{i \in \mathbb{N}}$ eventually reaches a maximum, that we denote with $\mathcal{P}_k^\infty(S)$.

Definition 2. A subset S of vertices is a *k-power dominating set* of G if $\mathcal{P}_k^\infty(S) = V(G)$. The minimum order of a power dominating set of G is the *k-power domination number* of G, denoted $\gamma_{\mathrm{P,k}}(G)$.

When $k = 0$, the definition corresponds to the normal domination parameter, while when $k = 1$, it coincides with the original power domination in graphs. Note that the problem is related to the problem of Zero forcing sets, as introduced in 2006 in [2].

Problems on general graphs

One first remark on generalized power domination is that for all $k \geq 0$, $\gamma_{\mathrm{P,k}}(G) \geq \gamma_{\mathrm{P,k+1}}(G)$. This generalizes the statement by Haynes et al. [12] that the power domination number of a graph is at most its domination number. As observed in [6], there is no hope to improve that bound, nor to find a upper bound on $\gamma_{\mathrm{P,k}}(G)$ in terms of $\gamma_{\mathrm{P,k+1}}(G)$. However, the following question remains of interest:

Question 1. *Can we find a characterization of the graphs such that $\gamma_{\mathrm{P,k}}(G) = \gamma_{\mathrm{P,\ell}}(G)$ for some $k < \ell$?*

Of course, the case when $\ell = k+1$ is very interesting, but more general characterization are also interesting. Observe that one possible construction for a graph satisfying this equality is to take any graph and one of its k-power dominating sets, S, and attach to every vertex v of S a subgraph on at least $\ell + 1$ vertices all of which are adjacent to v. Doing so, every ℓ-power dominating set would have to contain one vertex to dominate each of these subgraphs, thus S would also be an optimal ℓ-power dominating set. This motivates the following question, that captures somehow the interesting part of the previous question:

Question 2. *Can we find a characterization of the* 2-connected *graphs such that* $\gamma_{\mathrm{P,k}}(G) = \gamma_{\mathrm{P},\ell}(G)$ *for some* $k < \ell$*?*

Regular graphs

Regular graphs and especially cubic graphs are of special interest for domination. In a general study of regular graphs, the following conjecture was proposed in [8]:

Conjecture 3. *Let G be a connected r-regular graph on n vertices. If G is different from $K_{r,r}$, then*

$$\gamma_{\mathrm{P,k}}(G) \leq \frac{n}{r+1}$$

The bound proposed in that conjecture is straightforward for $r \leq k+1$ (since $\gamma_{\mathrm{P,k}}(G) = 1$, see [6]) and proved in [8] for $r = k+2$. The remaining cases are open. A simpler way of attacking the conjecture might be to consider random regular graphs, which have a rather predictable behaviour.

Question 4. *What can we say on the above conjecture on random regular graphs? Can we prove a similar bound?*

Planar graphs

Considering planar graphs in general, it is not possible to prove relevant bounds on the power domination number, as most constructions for tightening bounds are planar. However, it is possible to prove results on maximal planar graphs (triangulations of the plane). We managed to get the following:

Theorem 3 ([7])**.** *Every maximal planar graph on $n \geq 6$ vertices have (1-)power domination number at most $\frac{n-2}{4}$.*

The largest graph tightening the bound that we currently know is the triakis-tetrahedron, which has 10 vertices. There is no evidence that this bound is best possible, and it could probably be improved. The following question remains:

Question 5. *What is the best possible upper bound on the size of a minimum k-power dominating set of a maximal planar graph? To begin, what is the best α such that for all maximal planar graph G, $\gamma_{\mathrm{P},1}(G) \leq \alpha|V| + O(1)$?*

General constructions can be produced on $6k$ vertices that need at least k vertices to power dominate: for example, completing a family of disjoint (facial) octahedra into a maximal planar graph produces such a graph. Thus we know already $\frac{1}{6} \leq \alpha \leq \frac{1}{4}$. The same question can be asked for $\gamma_{\mathrm{P,k}}(G)$ for larger k. Constructions can be made of maximal planar graphs that need $\frac{n}{7}$ vertices at least to 2-power dominate, $\frac{n}{9}$ vertices at least to 3-power dominate. . . .

On an algorithmic point of view, we know that power domination is NP-complete even restricted to chordal graphs [12] and to planar bipartite graphs [5]. On the other hand, a polynomial time approximation scheme has long been known for planar graphs [3].

Question 6. *Is power domination NP-hard on maximal planar graphs? How can power domination be approximated on planar graphs?*

Special families

Finally, a natural question is to consider power domination on special families of graphs, in particular on lattices and on recursively defined families. In that context, an upper bound may easily be proved, simply by exhibiting a power dominating set. However, proving a lower bound is often much more tricky, and need clever techniques. Moreover, all power dominating set are not equivalent: some require longer time for propagation than others. In [9], the *propagation radius* of a graph was introduced, that can be defined as follows:

Definition 4. *The radius of a k-power dominating set S of a graph G is defined by*

$$\mathrm{rad}_{\mathrm{P},k}(G,S) = 1 + \min\{i:\ \mathcal{P}^i_{G,k}(S) = V(G)\}\,.$$

*The k-*propagation radius *of a graph G can be expressed as*

$$\mathrm{rad}_{\mathrm{P},k}(G) = \min\{\mathrm{rad}_{\mathrm{P},k}(G,S),\ S \text{ is a } k\text{-PDS of } G,\ |S| = \gamma_{\mathrm{P,k}}(G)\}\,.$$

Computing the propagation radius of a graph necessitates not only to find out what is the minimum power dominating set, but also to find such a set that propagates as fast as possible to the graph. This involves a much better understanding of what an efficient power dominating set is in the graph, and thus seems a relevant parameter to consider on graph families

Graph products were considered for the power domination, in particular the products of paths in [11, 10] and of paths and cycle in [4]. Those early studies did not take into account neither generalized power domination, nor the propagation radius, and they did not even settle all the products of paths. So it could be extended, and products of more than two paths should be considered:

Question 7. *Can we establish the k-power domination number and propagation radius for some products of simple graph families?*

In particular, we still wander whether something nice can be said on the hypercube. A wild guess would be that maybe $\gamma_{\mathrm{P,k+1}}(Q_{i+1}) = \gamma_{\mathrm{P,k}}(Q_i)$, but that is not true for $k = 1$. Pai and Chiu [13] showed that $\gamma(Q_5) = 7$ while $\gamma_{\mathrm{P,1}}(Q_6) = 6$, disproving the previous inequality. Though, there are no examples disproving the inequality for larger k. So the following question remains open:

Question 8. *Is it true that for $k \geq 1$, $\gamma_{\mathrm{P,k+1}}(Q_{i+1}) = \gamma_{\mathrm{P,k}}(Q_i)$? And if not, what is the first counterexample for a given k?*

A more general question, that sounds promising, would be to see whether the product and the propagation parameter k are related in some sense. The most general setting of the question would be the following:

Question 9. *For some product $\otimes$, can we find some non trivial way to relate $\gamma_{\mathrm{P,k}}(G)$, $\gamma_{\mathrm{P},\ell}(H)$ and some $\gamma_{\mathrm{P,f(k,}\ell)}(G \otimes H)$?*

References

[1] A. Aazami. Domination in graphs with bounded propagation: algorithms, formulations and hardness results. *J. Comb. Optim.*, 19(4):429–456, 2010.

[2] AIM Minimum Rank Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428(7):1628 – 1648, 2008.

[3] B. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. ACM*, 41(1):153–180, 1994.

[4] R. Barrera and D. Ferrero. Power domination in cylinders, tori, and generalized Petersen graphs. *Networks*, 58(1):43–49, 2011.

[5] D. J. Brueni and L. S. Heath. The PMU placement problem. *SIAM J. Discrete Math.*, 19(3):744–761, 2005.

[6] G. J. Chang, P. Dorbec, M. Montassier, and A. Raspaud. Generalized power domination of graphs. *Discrete Appl. Math.*, 160(12):1691–1698, 2012.

[7] P. Dorbec, A. Gonzales, and C. Pennarun. Power domination in maximal planar graphs. Manuscript.

[8] P. Dorbec, M. A. Henning, C. Löwenstein, M. Montassier, and A. Raspaud. Generalized power domination in regular graphs. *SIAM J. Discrete Math.*, 27(3):1559–1574, 2013.

[9] P. Dorbec and S. Klavžar. Generalized power domination: propagation radius and Sierpiński graphs. *Acta Appl. Math.*, 134:75–86, 2014.

[10] P. Dorbec, M. Mollard, S. Klavžar, and S. Špacapan. Power domination in product graphs. *SIAM J. Discrete Math.*, 22(2):554–567, 2008.

[11] M. Dorfling and M. A. Henning. A note on power domination in grid graphs. *Discrete Appl. Math.*, 154(6):1023–1027, 2006.

[12] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and M. A. Henning. Domination in graphs applied to electric power networks. *SIAM J. Discrete Math.*, 15(4):519–529, 2002.

[13] K.-J. Pai and W.-J. Chiu. A note on "on the power dominating sets of hypercubes". In *The 29th Workshop on Combinatorial Mathematics and Computation Theory*, 2012.

Contributed Problems

Determining the maximal number of k-dominating independent subsets in n-vertex graphs

by Zoltán Lóránt Nagy

Let $G = G(V, E)$ be a simple graph. For any vertex $v \in V(G)$, $d(v)$ denotes the degree of v, $N(v)$ denotes the set of neighbors of v, and $N[v]$ denotes the closed neighborhood, i.e., $N[v] := N(v) \cup \{v\}$. A set D of vertices in a graph G is k-dominating if each vertex in $V(G) \setminus D$ is adjacent to at least k vertices of D. We would like to study the maximum number of k-dominating independent sets (for brevity, k-DISes) in n-vertex graphs.
While the case $k = 1$, namely the case of maximal independent sets - which is originated from Erdős and Moser - is widely investigated, much less is known in general.

Our principal function is formulated in the following

Notation 1. *Let* $\mathrm{mi}_k(n)$ *denote the maximum number of k-DISes in graphs of order n, and let* $\mathrm{mi}_k(G)$ *denote the number of k-DISes in a graph G.*

Notation 2. *Let* $\zeta_k(G) := \sqrt[n]{\mathrm{mi}_k(G)}$ *for a fixed graph G on n vertices and let*

$$\zeta_k(n) := \sqrt[n]{\mathrm{mi}_k(n)}.$$

Some basic observations:

Observation 3. *(i)* $\zeta_k(n) \in [1, 2] \quad \forall k, n \in \mathbb{Z}^+, k \leq n.$

(ii) $\zeta_k(G) \leq \liminf \zeta_k(n) \quad \forall k \in \mathbb{Z}^+$ *and for every fixed graph G.*

(iii) $\forall k \quad \exists \lim_{n\to\infty} \zeta_k(n)$. *For brevity, we will use the notation* $\zeta_k := \lim_{n\to\infty} \zeta_k(n)$.

We only mention here that (ii) is based on the observation that for tG, i.e., t disjoint copies of a certain graph G, we have $\zeta_k(G) = \zeta_k(tG)$.

It was proved that the maximum number of k-dominating independent sets in n-vertex graphs is between $c_k \cdot \sqrt[2k]{2}^{\,n}$ and $c'_k \cdot \sqrt[k+1]{2}^{\,n}$ if $k \geq 2$, moreover the maximum number of 2-dominating independent sets in n-vertex graphs is between $c \cdot 1.22^n$ and $c' \cdot 1.246^n$; more precisely

Theorem 4. *The order of magnitude of the maximum number of 2-DISes is bounded as follows.*

$$1.22 < \sqrt[9]{6} \leq \zeta_2 \leq \sqrt[5]{3} < 1.2457.$$

Conjecture 1. *The lower bound in Theorem 4 is sharp, that is,* $\zeta_2(n) = \zeta_2(K_3 \square K_3)$ *for the Cartesian product of two triangles.*

In fact, it might be true that the following graphs are the best possible:

Construction 5. *Let n be large enough, and let*

$$G_n = \alpha K_3 \square K_3 + \beta K_4 \square K_4, \quad \textit{with} \;\; \alpha, \beta \in \mathbb{N}, \beta \leq 8.$$

(Observe that α and β is uniquely determined.)

Theorem 6. *For every $k > 3$,*

$$\sqrt[2k]{2} \leq \zeta_k \leq \sqrt[k+1]{2}.$$

Conjecture 2. *The lower bound in Theorem 6 is sharp if $k > 3$.*

Concerning $k = 3$, we conjecture that the Turán graph $T_{3\cdot3,3}$ provides the order of magnitude, as $\zeta_3(T_{9,3})\sqrt[9]{3} \leq \zeta_3(n)$. For larger value of k, we have $\zeta_k(K_{k,k}) = \sqrt[2k]{2} \leq \zeta_k$.

The aim is to make progress on Conjecture 1 and Conjecture 2.

Z. L. Nagy, On the number of k-dominating independent sets, Journal of Graph Theory, 84(4), (2017) 566–580.

Cranes and acyclic domination number

by Dömötör Pálvölgyi

In construction sites, the height of cranes must be chosen so that each of them can turn around in a full circle. This means that every crane can have in each range only one crane that is taller. First, suppose that all cranes have equal radius and consider the discrete model where the locations of the site are the vertices of a graph such that there is an edge between two locations if they are in each other's radius. (Note that not every graph can be derived this way.) It is not hard to see that the minimum number of cranes that need to be placed to cover each location equals the *acyclic domination number*, $\gamma_\emptyset$, of the graph, which is defined as the size of the smallest acyclic dominating set. For any graph, we have $\gamma \leq \gamma_\emptyset \leq \gamma_i$, where γ is the domination number and γ_i is the independent domination number. I propose to prove various bounds on $\gamma_\emptyset$.

If the cranes are not assumed to have equal radius, but at each location we can only place a crane of given radius, we get the natural directed version of the above graph problem. From a practical point of view, however, it seems more natural to study the geometric variants of the problem. For example, given a square and a fixed collection of cranes with various radius and height, can we cover the whole square with such cranes? What if we can change/decrease the heights?

Distance-l domination in hypergraphs

by Máté Vizer

Despite that domination is a well-investigated notion in graph theory, domination in hypergraphs is a relatively new subject. It was introduced in [1]; for more recent results and references see [2, 3]. The main aim of this project would be to investigate hypergraph domination notions analogous to the graph domination ones. In [3] with Bujtás, Patkós and Tuza we started such a research concerning distance-l domination in hypergraphs.

Distance-l domination in hypergraphs

In *distance-l domination* a vertex v dominates all vertices that are at distance at most l from v. As the definition of distance in graphs involves paths, and paths in hypergraphs can be defined in several ways, distance-l domination could be addressed with each of those definitions, however only so-called 'Berge paths' offer new problems in our context.

A *Berge path* of length l is a sequence $v_0, H_1, v_1, H_2, v_2, \dots, H_l, v_l$ with $v_i \in V(\mathcal{H})$ for $i = 0, 1, ..., l$ and $v_{i-1}, v_i \in H_i \in \mathcal{E}(\mathcal{H})$ for $i = 1, 2, ..., l$. The distance $d_{\mathcal{H}}(u, v)$ of two vertices $u, v \in V(\mathcal{H})$ is the length of a shortest Berge path from u to v. The *ball centered at u and of radius l* consists of those vertices of $\mathcal{H}$ which are at distance at most l from u; it will be denoted by $B_l(u)$. We call $D \subset V(\mathcal{H})$ a *distance-l dominating set* of $\mathcal{H}$ if $\bigcup_{u \in D} B_l(u) = V(\mathcal{H})$. Equivalently we can say that $D \subset V(\mathcal{H})$ is a distance-l dominating set if and only if $D \cap B_l(v) \neq \emptyset$ for all $v \in V(\mathcal{H})$. Note that distance-1 dominating sets are the usual dominating sets. The minimum size of a distance-l dominating set in a hypergraph $\mathcal{H}$ is the distance-l domination number $\gamma_d(\mathcal{H}, l)$ and let us denote by $\gamma_{dc}(\mathcal{H}, l)$ the analogous notion for connected hypergraphs.

For $k \geq 2$ and $l, \gamma \geq 1$ let $n_{dc}(k, \gamma, l)$ denote the minimum number of vertices that a k-uniform **connected** hypergraph $\mathcal{H}$ must contain if it has $\gamma_d(\mathcal{H}, l) \geq \gamma$. (Note that the problem is almost trivial if we do not suppose that the hypergraph is connected.)

To state our main result concerning $n_{dc}(k, \gamma, l)$ we need to define the following function:

$$f(k, \gamma, l) := \begin{cases} \frac{l}{2} k\gamma + \max\{k, \gamma\} & \text{if } l \text{ is even,} \\ \frac{l+1}{2} k\gamma & \text{if } l \text{ is odd.} \end{cases}$$

Theorem 1. *For any $k \geq 2$, $l \geq 4$ and $\gamma \geq 3$ we have*

$$k \left\lceil \left(\frac{l-1}{2} - 1 \right) \gamma \right\rceil < n_{dc}(k, \gamma, l) \leq f(k, \gamma, l).$$

Or we can state it in "Meier, Moon [4] style":

Theorem 2. *If $\mathcal{H}$ is a connected k-uniform hypegraph with $|V(\mathcal{H})| = n$ and $l > 4$, then*

$$\gamma_{dc}(\mathcal{H}, l) \leq \frac{n}{k} \cdot \frac{2}{l-3}.$$

Problem 1. *Close the gap of roughly $2k\gamma$ between the upper and lower bounds in Theorem 1.*

References

[1] B. D. Acharya, Domination in hypergraphs, *AKCE International Journal of Graphs and Combinatorics*, **4**(2) (2007), 117–126.

[2] Cs. Bujtás, M. A. Henning, and Zs. Tuza, Transversals and domination in uniform hypergraphs, *European Journal of Combinatorics*, **33**(1) (2012), 62–71.

[3] Cs. Bujtás, B. Patkós, Zs. Tuza, and M. Vizer, The minimum number of vertices in uniform hypergraphs with given domination number, *Discrete Mathematics*, (2017), to appear.

[4] A. Meir and J. W. Moon, *Relations between packing and covering number of a tree*, Pacific Journal of Mathematics, 61 (1975), pp. 225–233.