Splitting noncommutative dynamical systems in almost periodic and weakly mixing parts

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State preserving C^*-dynamical systems

Let us denote for a C^*-algebra A by $End(A)$ the semigroup of all $*$-endomorphisms of A and by $Aut(A)$ the group of all $*$-automorphisms of A.

A *state preserving C^*-dynamical system* is a quadruple (A, S, α, φ) consisting of a C^*-algebra A, a semigroup S, a semigroup-homomorphism

$$\alpha : S \ni s \longmapsto \alpha_s \in End(A),$$

and a state φ of A, such that the invariance condition

$$\varphi \circ \alpha_s = \varphi, \qquad s \in S.$$

is satisfied. We shall always assume that S is unital and α maps the unit of S into the identity automorphism of A.

If A is a W^*-algebra, the $*$-endomorphisms α_s are normal and φ is normal state, then we call

$(A\,,S\,,\alpha\,,\varphi)$ a *state preserving W^*-dynamical system*.

Let $(A\,,S\,,\alpha\,,\varphi)$ be a state preserving C^*-dynamical system and let us consider the GNS-representation

$$\pi_\varphi : A \longrightarrow B(H_\varphi)$$

of φ with canonical cyclic vector ξ_φ. Then

$$U_s\pi_\varphi(x)\xi_\varphi = \pi_\varphi\Big(\alpha_s(x)\Big)\xi_\varphi\,, \qquad x \in A$$

defines a linear isometry $U_s : H_\varphi \longrightarrow H_\varphi$ and

$$S \ni s \longmapsto U_s$$

is a representation of S by linear isometries on H_φ satisfying

$$U_s\pi_\varphi(a) = \pi_\varphi\Big(\alpha_s(a)\Big)U_s\,, \qquad s \in S$$

and

$$U_s\xi_\varphi = \xi_\varphi\,, \qquad s \in S\,.$$

If S is a group then every α_s is a $*$-automorphism, $S \ni s \longmapsto U_s$ is a unitary representation and, for every $s \in S$ and $a \in A$,

$$\pi_\varphi\Big(\alpha_s(a)\Big) = U_s \pi_\varphi(a) U_s^* = Ad(U_s)\pi_\varphi(a)\,. \quad (1)$$

Therefore, denoting by M the von Neumann algebra $\pi_\varphi(A)'' \subset B(H_\varphi)$ and by ω_{ξ_φ} the vector state

$$M \ni T \longmapsto (T\xi_\varphi|\xi_\varphi)\,,$$

we can define a state preserving W^*-dynamical system $(M\,, S\,, \beta\,, \omega_{\xi_\varphi})$ by putting

$$\beta_s = Ad(U_s)\,, \qquad s \in S\,.$$

According to (1), the GNS-representation π_φ yields a natural imbedding of the original C^*-dynamical system $(A\,, S\,, \alpha\,, \varphi)$ in the W^*-dynamical system $(M\,, S\,, \beta\,, \omega_{\xi_\varphi})$.

If S is not a group, it can happen that for some $a \in A$ and $s \in S$ we have $\pi_\varphi(a) = 0$ but

$\pi_\varphi\Big(\alpha_s(a)\Big) \neq 0$, so cannot exist state preserving W^*-dynamical system $(M\,,S\,,\beta\,,\omega_{\xi_\varphi})$ satisfying

$$\pi_\varphi\Big(\alpha_s(a)\Big) = \beta_s\Big(\pi_\varphi(a)\Big)$$

and therefore we cannot imbed $(A\,,S\,,\alpha\,,\varphi)$ in a W^*-dynamical system $(M\,,S\,,\beta\,,\omega_{\xi_\varphi})$ with an appropriate action β.

Nevertheless, if the support projection of φ in the second dual A^{**} belongs to the centre of A^{**}, what means that the cyclic vector ξ_φ is also separating for $\pi_\varphi(A)''$, then for arbitrary S we have a (uniquely defined) state preserving W^*-dynamical system $(M\,,S\,,\beta\,,\omega_{\xi_\varphi})$ satisfying

$$\pi_\varphi\Big(\alpha_s(a)\Big) = \beta_s\Big(\pi_\varphi(a)\Big)$$

for all $s \in S$ and $a \in A$. This happens, for example, if φ is a trace.

Moreover, if S is a topological semigroup and the two-point functions

$$S \ni s \longmapsto \varphi\Big(y\,\alpha_s(x)\Big)\,, \qquad x\,,y \in A$$

are continuous, then the isometry semigroup

$$S \ni s \longmapsto U_s$$

is continuous with respect to the weak operator topology, hence also with respect to the strong operator topology. It follows that the orbits

$$S \ni s \longmapsto \beta_s(T)\,, \qquad T \in M$$

are continuous with respect to the strong operator topology.

Almost periodicity and weakly mixing

Let $M \subset B(H)$ be a von Neumann algebra having a cyclic and separating vector ξ_o of unit lenght, S a locally compact unital semigroup and

$$\alpha : S \ni s \longmapsto \alpha_s$$

a unital semigroup homomorphism of S in the semigroup of all normal $*$-endomorphisms of M, which is continuous with respect to the pointwise strong operator topology and which leaves invariant ω_{ξ_o}:

$$\omega_{\xi_o} \circ \alpha_s = \omega_{\xi_o} , \qquad s \in S .$$

We define the strongly continuous isometry semigroup

$$S \ni s \longmapsto U_s$$

by

$$U_s x\xi_o = \alpha_s(x)\xi_o , \qquad x \in M .$$

We say that the dynamical system $(M\,,S\,,\alpha\,,\omega_{\xi_o})$ is *almost periodic* if all orbits

$$\{U_s\xi\,;\,s\in S\}\subset H\,,\qquad \xi\in H$$

are relatively norm-compact and we say that it is *weakly mixing* (to 0) if $\{0\}\subset H$ is the only relatively norm-compact orbit of U.

Weakly mixing is usually described by some equivalent convergence property. For example, in the case of the additive semigroup $S=\mathbb{N}$ of all integers $n\geq 0$, by a classical theorem of B. O. Koopman and John von Neumann (see (4))$(M\,,\mathbb{N}\,,\alpha\,,\omega_{\xi_o})$ is weakly mixing to 0 if and only if, for every $a_o\,,a_1\in M$,

$$\omega_{\xi_o}\Big(a_o\,\alpha_n(a_1)\Big)\longrightarrow 0$$

for $n\to\infty$ avoiding a zero density set of natural numbers, or equivalently,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Big|\omega_{\xi_o}\Big(a_o\,\alpha_k(a_1)\Big)\Big|=0\,. \qquad (2)$$

Similar descriptions are possible in the case of an amenable group S by replacing in (2) the sequence $\{0, 1, \ldots n-1\}, n \geq 1$, with a Følner sequence, the sum with the integral with respect to the Haar measure, and the denominators n with the Haar measure of the sets in the used Følner sequence.

In the general case we denote by H_{AP} the set of all $\xi \in H$ for which $\{U_s \xi\,;\, s \in S\}$ is relatively norm-compact. Then H_{AP} is a U-invariant closed linear subspace of H. Even more, we have $U\Big(H_{AP}\Big) = H_{AP}$, so also $H \ominus H_{AP}$ is U-invariant.

We notice that if $S \ni s \longmapsto U_s$ is just a strongly continuous bounded semigroup of bounded linear operators on H (so the operators U_s are not assumed to be isometries) then $H \ominus H_{AP}$ is not necessarily U-invariant. One can ask to find an U-invariant closed linear subspace of H such that H is the direct sum of H_{AP}

and this subspace. This is the subject of the splitting theorem of Konrad Jacobs (see (3)) which was extended to more general Banach spaces by Karel Deleeuw and Irwing Glicksberg (see (1)).

However there is no hope to split the von Neumann algebra M in a direct sum of α-invariant subalgebras and we have to look for an other kind of "splitting". In fact we can show (in the case of $S = \mathbb{N}$ done by C. Niculescu, A. Ströh and L. Zs., see (5) Theorem 4.2 and Proposition 5.5) that there exists a greatest α-invariant von Neumann subalgebra M^{AP} of M with the following properties:

- $M^{AP}\xi_o$ is a dense linear subspace of H_{AP};

- M^{AP} is the weak*-closed linear span of the union of all finite-dimensional α-invariant linear subspaces of M ;

- the restriction of every endomorphism α_s to M^{AP} is a $*$-automorphism of M^{AP};

- any α-invariant von Neumann subalgebra N of M , such that the state preserving dynamical system $(N\,,S\,,\alpha\,,\omega_{\xi_o}|N)$ is almost periodic, is contained in M^{AP};

- M^{AP} is left invariant by the modular automorphism group of the faithful normal state ω_{ξ_φ} of M . Thus there exists a faithful normal conditional expectation E of M onto M^{AP} which leaves invariant the state ω_{ξ_o} and commutes with the action α .

Instead of a weakly mixing factor we have a relative weakly mixing property which can be described by replacing the state ω_{ξ_o} with the conditional expectation $E : M \longrightarrow M^{AP}$. For example, in the case of the additive semigroup $S = \mathbb{N}$ we have for every $a_o, a_1 \in M$

$$E\Big(a_o \alpha_n(a_1) - E(a_o)\,\alpha_n\big(E(a_1)\big)\Big) \longrightarrow 0$$

with respect to the weak operator topology for $n \to \infty$ avoiding a zero density set of natural numbers. In many cases (but not always) the above convergence holds with respect to the strong operator topology. This happens, for example, if M is commutative, in which case one can prove (H. Furstenberg, see (2) Theorem 8.3) that for every integer $p \geq 1$ and every $a_o, a_1, \dots, a_p \in M$ we have

$$\begin{aligned} E\Big(&a_o \alpha_n(a_1) \dots \alpha_{pn}(a_p) \\ &- E(a_o)\,\alpha_n\big(E(a_1)\big) \dots \alpha_{pn}\big(E(a_p)\big)\Big) \longrightarrow 0 \end{aligned}$$

with respect to the strong operator topology for $n \to \infty$ avoiding a zero density set of natural numbers.

Proof idea

Let $(M\,,S\,,\alpha\,,\omega_{\xi_o})$ be a state preserving W^*-dynamical system as above.

First we prove that, for every weak*-closed linear subspace N of M satisfying $\overline{N'\xi_o} = H$ and $N\xi_o \subset H_{AP}$, the set $\mathcal{G}$ of all linear contractions $\Theta : N \longrightarrow N$ for which

$$\Theta(T)\xi_o \in \overline{\{U_s T\xi_o\,;\; s \in S\}}\,, \qquad T \in N\,,$$

endowed with the topology of the pointwise strong operator convergence, is a compact topological group with respect to composition, having the identical map of N as neutral element. Moreover, every $\Theta \in \mathcal{G}$ is weak*-continuous and $\mathcal{G}$ has the following recurrence property:

For every integer $p \geq 1$,

$$\Theta_1, \ldots \Theta_p \in \mathcal{G}\,, T_1, \ldots T_p \in N\,, \xi_1, \ldots \xi_p \in H$$

and $\varepsilon > 0$, there exists a relatively dense (= syndetic) set $\mathcal{N} \subset \mathbb{N}$ such that

$$\Big\| \Theta_j^n(T_j)\xi_j - T_j\xi_j \Big\| \leq \varepsilon\,, \qquad 1 \leq j \leq p\,, n \in \mathcal{N}\,.$$

Afterwards, we define

$$M_{AP} = \Big\{ x \in M\,;\, x\xi_o \in H_{AP} \Big\}$$

and apply the above statement to $N = M_{AP}$. It turns out that every α_s belongs to $\mathcal{G}$ and we can use facts from the theory of Banach space representations of compact groups.

An application

Let $(M\,,\mathbb{N}\,,\alpha\,,\omega_{\xi_o})$ be a state preserving C^*-dynamical system as above and let us assume that M is commutative.

For every $0 \leq a \in M$ and $\varepsilon > 0$ there exists a zero density set $\mathcal{D} \subset \mathbb{N}$ and an integer $n_\varepsilon \geq 1$ such that

$$\begin{aligned} &\Big|\omega_{\xi_o}\Big(a\,\alpha_n(a)\,...\,\alpha_{pn}(a)\Big) \\ &\quad - \omega_{\xi_o}\bigg(E(a)\,\alpha_n\Big(E(a)\Big)\,...\,\alpha_{pn}\Big(E(a)\Big)\bigg)\bigg| \\ = &\bigg|\omega_{\xi_o}\bigg(E\bigg(a\,\alpha_n(a)\,...\,\alpha_{pn}(a) \\ &\quad - E(a)\,\alpha_n\Big(E(a)\Big)\,...\,\alpha_{pn}\Big(E(a)\Big)\bigg)\bigg)\bigg| \leq \frac{\varepsilon}{2} \end{aligned}$$

for $n_\varepsilon \leq n \notin \mathcal{D}$. Since $E(a) \in M_{AP}$ and α is almost periodic on M_{AP}, there exists a relatively dense (= syndetic) set $\mathcal{N} \subset \mathbb{N}$ such that

$$\left|\omega_{\xi_o}\Big(E(a)\,\alpha_n\big(E(a)\big)\ldots\alpha_{pn}\big(E(a)\big)\Big)\right.$$
$$\left.-\,\omega_{\xi_o}\Big(E(a)^p\Big)\right| \leq \frac{\varepsilon}{2}$$

for $n \in \mathcal{N}$. Consequently

$$\left|\omega_{\xi_o}\big(a\,\alpha_n(a)\ldots\alpha_{pn}(a)\big)-\omega_{\xi_o}\Big(E(a)^p\Big)\right| \leq \varepsilon$$

and so

$$\omega_{\xi_o}\big(a\,\alpha_n(a)\ldots\alpha_{pn}(a)\big) \geq \omega_{\xi_o}\Big(E(a)^p\Big)-\varepsilon$$

for all $n_\varepsilon \leq n \in \mathcal{N}\setminus\mathcal{D}$. We notice that the set $\{n \in \mathcal{N}\setminus\mathcal{D}\,;n \geq n_\varepsilon\}$ is of strictly positive lower density.

In particular, if $0 \leq a \in M$ and $a \neq 0$, then

$$a\,\alpha_n(a)\ldots\alpha_{pn}(a) \neq 0$$

for all n belonging to a natural number set of strictly positive lower density. This is the ergodic theoretical version of E. Szemerédi's celebrated heorem on arithmetical progressions solving a conjecture of P. Erdős and P. Turán (see (6) and (2)).

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