Summability of multi-dimensional Fourier series

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1. Partial sums of Fourier series

The $L_p(\mathbb{T})$ space is equipped with the norm (or quasi-norm)

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{T}} |f|^p \, d\lambda \right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{T}} |f|, & p = \infty, \end{cases}$$

where $\mathbb{T} := [-\pi, \pi]$.

The Fourier coefficients and partial sums of the Fourier series of $f \in L_1(\mathbb{T})$ is defined by

$$\widehat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-\imath k x} \, dx \qquad (k \in \mathbb{Z}),$$

$$s_n f(x) := \sum_{|k| \le n} \widehat{f}(k) e^{\imath k x} \qquad (n \in \mathbb{N}).$$

Theorem 1 (Riesz). *If* $f \in L_p(\mathbb{T})$ *for some* $1 < p < \infty$ *then*

$$\|s_n f\|_p \le C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{in } L_p\text{-norm.}$$

Theorem 2 (Carleson, Hunt). *If* $f \in L_p(\mathbb{T})$ *for some* $1 < p < \infty$ *then*

$$\left\| \sup_{n\in\mathbb{N}} |s_n f| \right\|_p \le C_p \|f\|_p$$

and if $1 < p \le \infty$ *then*

$$\lim_{n\to\infty} s_n f = f \qquad a.e.$$

2. Summability of one-dimensional Fourier series

The Fejér and Riesz means are given by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) = \sum_{|j|\le n}\left(1-\frac{|j|}{n}\right)\widehat{f}(j)e^{\imath jx}$$

and

$$\sigma_n^\alpha f(x) := \sum_{|j|\le n}\left(1-\left(\frac{|j|}{n}\right)^\gamma\right)^\alpha \widehat{f}(j)e^{\imath jx}$$

with $0<\alpha<\infty, 1\le\gamma<\infty$. Let

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f| .$$

Corollary 1 (Zygmund, Riesz). *If* $0 < \alpha < \infty$ *and* $f \in L_1(\mathbb{T})$ *then*

$$\|\sigma_*^\alpha f\|_{1,\infty} = \sup_{\rho>0} \rho\, \lambda\,(\sigma_*^\alpha f > \rho) \le C\,\|f\|_1\,.$$

This weak type $(1,1)$ inequality and the density argument of Marcinkiewicz and Zygmund imply

Corollary 2 (Lebesgue, Riesz). *If* $0 < \alpha < \infty$ *and* $f \in L_1(\mathbb{T})$ *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad a.e.$$

Theorem 3 (Marcinkiewicz, Zygmund). *Suppose that $X_0 \subset L_p$ is dense in L_p. Let T_n $(n \in \mathbb{N})$ be linear operators such that*

$$\lim_{n\to\infty} T_n f = f \quad \text{a.e. for every } f \in X_0.$$

If

$$\sup_{\rho>0} \rho\, \lambda\,(T_* f > \rho)^{1/p} \leq C\, \|f\|_p \qquad (f \in L_p)$$

for some $1 \leq p < \infty$, then

$$\lim_{n\to\infty} T_n f = f \quad \text{a.e. for every } f \in L_p,$$

where

$$T_* f := \sup_{n\in\mathbb{N}} |T_n f| \qquad (f \in L_p).$$

3. More-dimensional partial sums

Let

$$u \cdot x := \sum_{k=1}^{d} u_k x_k, \qquad \|x\|_q := \begin{cases} \left(\sum_{k=1}^{d} |x_k|^q\right)^{1/q}, & 0 < q < \infty; \\ \sup_{i=1,\ldots,d} |x_i|, & q = \infty. \end{cases}$$

Taking the Kronecker product of d trigonometric systems we obtain the d-dimensional trigonometric system:

$$e^{\imath k \cdot x} = \prod_{j=1}^{d} e^{\imath k_j x_j}.$$

The Fourier coefficients of an integrable function are given by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath k \cdot x} \, dx \qquad (k \in \mathbb{Z}^d).$$

For $f \in L_1(\mathbb{T}^d)$ the nth *ℓ_q-partial sum* ($q = 1$ triangular, $q = 2$ circular, $q = \infty$ cubic partial sum) $s_n^q f$ ($n \in \mathbb{N}$) is given by

$$s_n^q f(x) := \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} \widehat{f}(k) e^{\imath k \cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) D_n^q(u)\, du$$

where

$$D_n^q(u) := \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} e^{\imath k \cdot u}$$

is the ℓ_q *Dirichlet kernel.*

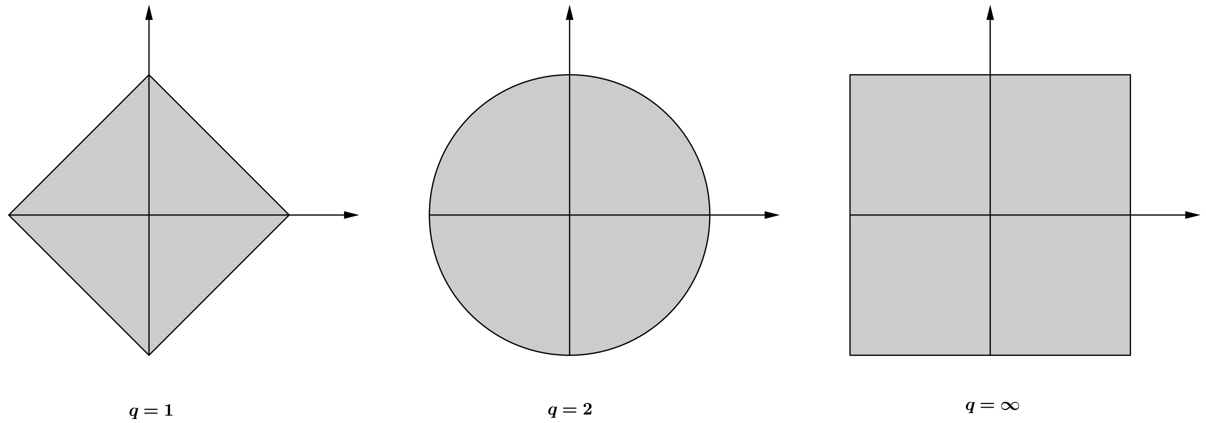

FIGURE 1. Regions of the ℓ_q-partial sums for $d = 2$.

Theorem 4 (Fefferman). *If* $q=1,\infty$ *and* $f\in L_p(\mathbb{T}^d)$ *for some* $1<p<\infty$ *then*

$$\|s_n^q f\|_p \le C_p \|f\|_p \qquad (n\in\mathbb{N})$$

and

$$\lim_{n\to\infty} s_n^q f = f \qquad \textit{in } L_p\textit{-norm.}$$

If $q=2$ *then the same result is valid only for* $p=2$.

Theorem 5 (Fefferman). *If* $q=1,\infty$ *and* $f\in L_p(\mathbb{T}^d)$ *for some* $1<p<\infty$ *then*

$$\left\|\sup_{n\in\mathbb{N}} |s_n^q f|\right\|_p \le C_p \|f\|_p$$

and if $1<p\le\infty$, *then*

$$\lim_{n\to\infty} s_n^q f = f \qquad a.e.$$

Theorem 5 does not hold for circular partial sums.

Theorem 6 (Stein and Weiss)**.** *If* $q = 2$ *and* $p < 2d/(d+1)$*, then there exists a function* $f \in L_p(\mathbb{T}^d)$ *whose circular partial sums* $s_n^q f$ *diverge almost everywhere. For* $p = 2$ *it is an open problem.*

4. ℓ_q-SUMMABILITY

The ℓ_q *Fejér and Riesz means* of f are defined by

$$\begin{aligned} \sigma_n^q f(x) &= \sum_{k\in\mathbb{Z}^d, \|k\|_q\le n} \left(1-\frac{\|k\|_q}{n}\right)\widehat{f}(k)e^{\imath k\cdot x} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-u)K_n^q(u)\,du, \end{aligned}$$

$$\begin{aligned} \sigma_n^{q,\alpha} f(x) &= \sum_{k\in\mathbb{Z}^d, \|k\|_q\le n} \left(1-\left(\frac{\|k\|_q}{n}\right)^{\gamma}\right)^{\alpha}\widehat{f}(k)e^{\imath k\cdot x} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-u)K_n^{q,\alpha}(u)\,du, \end{aligned}$$

where the ℓ_q *Fejér- and Riesz kernels* are given by

$$K_n^q(u) := \sum_{k\in\mathbb{Z}^d, \|k\|_q \le n} \left(1 - \frac{\|k\|_q}{n}\right) e^{\imath k\cdot u}$$

and

$$K_n^{q,\alpha}(u) := \sum_{k\in\mathbb{Z}^d, \|k\|_q \le n} \left(1 - \left(\frac{\|k\|_q}{n}\right)^\gamma\right)^\alpha e^{\imath k\cdot u}.$$

Then

$$\sigma_n^q f(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_k^q f(x) \qquad (q = 1, q = \infty).$$

The nth divided difference of a function f is introduced as

$$[x_1]f := f(x_1), \qquad [x_1,\ldots,x_n]f := \frac{[x_1,\ldots,x_{n-1}]f - [x_2,\ldots,x_n]f}{x_1 - x_n}.$$

It is known that

$$D_n^1(x) = [\cos x_1,\ldots,\cos x_d]G_n, \qquad (x \in \mathbb{T}^d),$$

where

$$G_n(\cos x) := (-1)^{[(d-1)/2]} 2\cos(x/2)(\sin x)^{d-2} \operatorname{soc}\left((n+1/2)x\right)$$

and

$$\operatorname{soc} x := \begin{cases} \cos x, & \text{if } d \text{ is even;} \\ \sin x, & \text{if } d \text{ is odd.} \end{cases}$$

If $d = 1$ then

$$D_n^1(x) = D_n^q(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)},$$

if $d = 2$ then

$$\begin{aligned} D_n^1(x) &= [\cos x_1, \cos x_2]G_n = \frac{[\cos x_1]G_n - [\cos x_2]G_n}{\cos x_1 - \cos x_2} \\ &= 2\frac{\cos(x_1/2)\cos((n+1/2)x_1) - \cos(x_2/2)\cos((n+1/2)x_2)}{\cos x_1 - \cos x_2}. \end{aligned}$$

The cubic Dirichlet kernels can be given by

$$D_n^\infty(x) = \prod_{i=1}^{d} \frac{\sin((n+1/2)x_i)}{\sin(x_i/2)}.$$

If $q = 2$ then the continuous version of the Dirichlet kernel

$$D_t^2(x) := \int_{\mathbb{R}^d} \mathbf{1}_{\{\|v\|_2 \leq t\}} e^{\imath x \cdot v} \, dv$$

can be expressed as

$$D_t^2(x) = \|x\|_2^{-d/2} t^{d/2} J_{d/2}\left(2\pi \|x\|_2 t\right),$$

where

$$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_{-1}^{1} e^{\imath ts} (1-s^2)^{v-1/2} \, ds \qquad (\nu > -1/2, t > 0)$$

are the *Bessel functions.*

5. Norm convergence of the summability

A Banach space B consisting of measurable functions on $\mathbb{T}^d$ is called a *homogeneous Banach space*, if

(i) $\|f\|_1 \le C\|f\|_B$ for all $f \in B$,

(ii) for all $f \in B$ and $x \in \mathbb{T}^d$, $T_x f := f(\cdot - x) \in B$ and $\|T_x f\|_B = \|f\|_B$,

(iii) the function $x \mapsto T_x f$ from $\mathbb{T}^d$ to B is continuous for all $f \in B$.

It is easy to see that the spaces $L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$, $C(\mathbb{T}^d)$, Lorentz spaces $L_{p,q}(\mathbb{T}^d)$ $(1 < p < \infty, 1 \le q < \infty)$ and Hardy space $H_1(\mathbb{T}^d)$ are homogeneous Banach spaces.

Theorem 7. *If* $q = 1, \infty$ *and* $\alpha > 0$ *then*

$$\int_{\mathbb{T}^d} |K_n^{q,\alpha}(x)| \, dx \leq C \qquad (n \in \mathbb{N}).$$

If $q = 2$ *then the same holds for* $\alpha > (d-1)/2$.

Theorem 8. *If* $q = 1, \infty$, $\alpha > 0$ *and* B *is a homogeneous Banach space on* $\mathbb{T}^d$ *then*

$$\|\sigma_n^{q,\alpha} f\|_B \leq C \, \|f\|_B \qquad (n \in \mathbb{N})$$

and

$$\lim_{n \to \infty} \sigma_n^{q,\alpha} f = f \qquad \text{in } B\text{-norm for all } f \in B.$$

If $q = 2$ *then the same holds for* $\alpha > (d-1)/2$.
The inequality holds also for $B = L_\infty(\mathbb{T}^d)$.

6. Circular Bochner-Riesz means ($q = \gamma = 2$)

Theorem 9. *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 \leq \alpha \leq (d-1)/2$ and*

$$p \leq \frac{2d}{d+1+2\alpha} \qquad \text{or} \qquad p \geq \frac{2d}{d-1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are not uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 2).

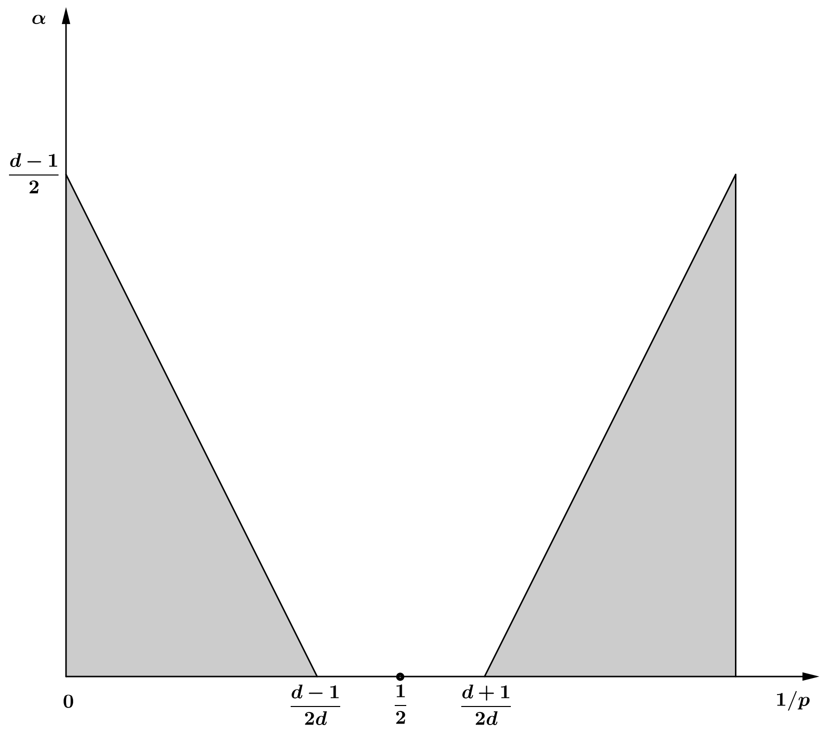

FIGURE 2. Uniform unboundedness of $\sigma_n^{2,\alpha}$.

Theorem 10 (Carleson and Sjölin). *Suppose that* $d = 2$ *and* $q = \gamma = 2$. *If* $0 < \alpha \leq 1/2$ *and*

$$\frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ *are uniformly bounded on* $L_p(\mathbb{T}^d)$ *(see Figure 3).*

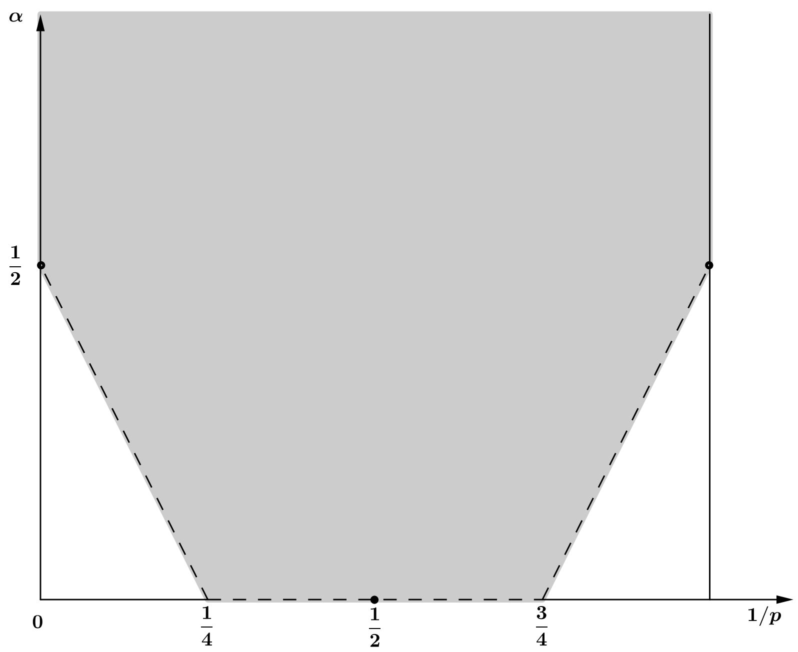

FIGURE 3. Uniform boundedness of $\sigma_n^{2,\alpha}$ when $d = 2$.

Theorem 11 (Fefferman). *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $\frac{d-1}{2(d+1)} \leq \alpha \leq \frac{d-1}{2}$ and*

$$\frac{2d}{d+1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 4).

Theorem 12 (Stein and Weiss). *Suppose that $d \geq 3$ and $q = \gamma = 2$. If $0 < \alpha < \frac{d-1}{2(d+1)}$ and*

$$\frac{2(d-1)}{d-1+4\alpha} < p < \frac{2(d-1)}{d-1-4\alpha},$$

then the Bochner-Riesz operators $\sigma_n^{2,\alpha}$ are uniformly bounded on $L_p(\mathbb{T}^d)$ (see Figure 4).

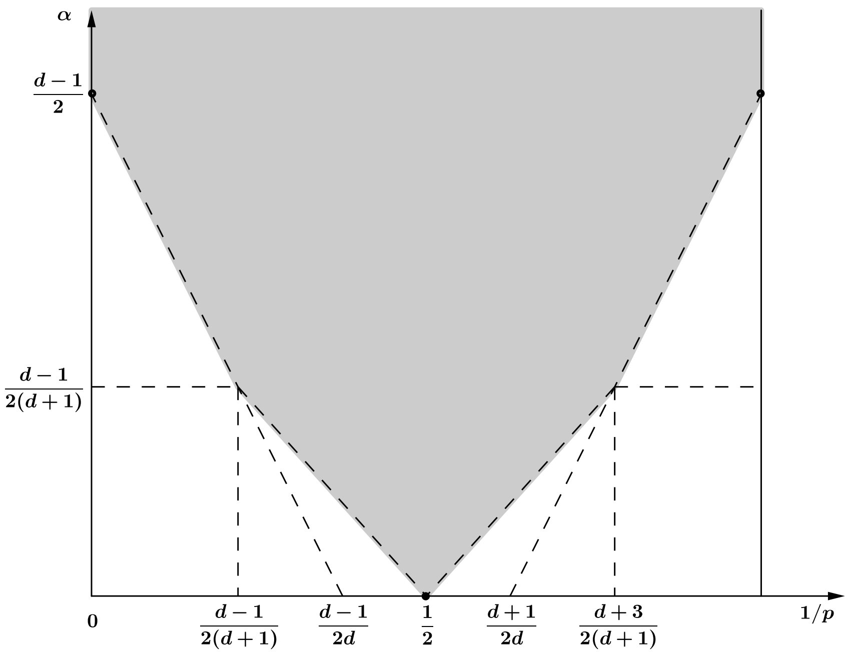

FIGURE 4. Uniform boundedness of $\sigma_n^{2,\alpha}$ when $d \geq 3$.

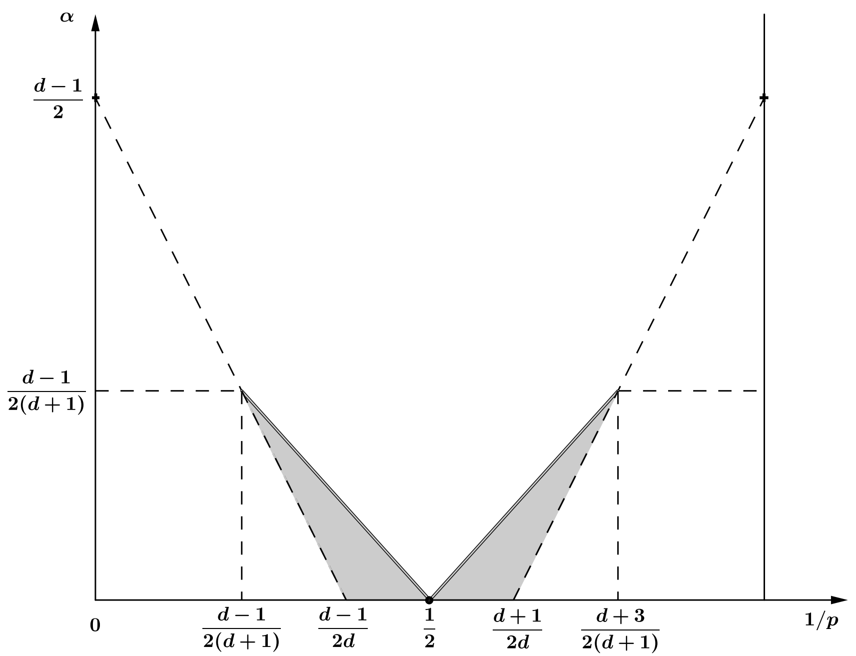

FIGURE 5. Open question of the uniform boundedness of $\sigma_n^{2,\alpha}$ when $d \geq 3$.

7. $H_p(\mathbb{T}^d)$ Hardy spaces

A function f is in the periodic Hardy space $H_p(\mathbb{T}^d)$ $(0 < p \leq \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{0<t} \left| f * P_t^d \right| \right\|_p < \infty,$$

where

$$P_t^d(x) := \sum_{m \in \mathbb{Z}^d} e^{-t\|m\|_2} e^{2\pi \imath m \cdot x} \qquad (x \in \mathbb{T}^d, t > 0)$$

is the d-dimensional *Poisson kernel*. Then

$$H_p(\mathbb{T}^d) \sim L_p(\mathbb{T}^d) \qquad (1 < p \leq \infty).$$

The atomic decomposition is a useful characterization of Hardy spaces. A bounded function a is a H_p-*atom* if there exists a cube $I \subset \mathbb{T}^d$ such that

$$
\begin{aligned}
&\operatorname{supp} a \subset I, \\
&\|a\|_\infty \le |I|^{-1/p}, \\
&\int_I a(x) x^k \, dx = 0 \text{ for all multi-indices } k = (k_1, \ldots, k_d) \\
&\qquad\qquad\qquad\qquad \text{with } \|k\|_2 \le [d(1/p - 1)].
\end{aligned}
$$

Theorem 13. *A function f is in $H_p(\mathbb{T}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of H_p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of distributions.}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$

Theorem 14. *For each* $n \in \mathbb{N}^d$, *let* $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ *be a bounded linear operator and let*

$$V_* f := \sup_{n \in \mathbb{N}^d} |V_n f|.$$

Suppose that

$$\int_{\mathbb{T}^d \setminus I} |V_* a|^{p_0} \, d\lambda \leq C_{p_0}$$

for all H_{p_0}*-atoms* a *and for some fixed* $0 < p_0 \leq 1$, *where the cube* I *is the support of the atom. If* V_* *is bounded from* $L_{p_1}(\mathbb{T}^d)$ *to* $L_{p_1}(\mathbb{T}^d)$ *for some* $1 < p_1 \leq \infty$, *then*

$$\|V_* f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d))$$

for all $p_0 \leq p \leq p_1$.

8. A.E. CONVERGENCE OF THE ℓ_q-SUMMABILITY

The *maximal operator* is defined by

$$\sigma_*^{q,\alpha} f := \sup_{n\in\mathbb{N}} \left|\sigma_n^{q,\alpha} f\right| .$$

If $f \in L_\infty(\mathbb{T}^d)$, then

$$\left\|\sigma_*^{q,\alpha} f\right\|_\infty \leq C \left\|f\right\|_\infty .$$

Moreover, if $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then

$$\left\|\sigma_*^{q,\alpha} f\right\|_p \leq C_p \left\|f\right\|_p .$$

Theorem 15 (Oswald, Weisz). *If* $q = 1, \infty$, $\alpha > 0$ *and* $d/(d+1) < p \le \infty$ *then*

$$\left\|\sigma_*^{q,\alpha} f\right\|_p \le C_p \left\|f\right\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d))$$

and for $f \in H_{d/(d+1)}(\mathbb{T}^d)$,

$$\left\|\sigma_*^{q,\alpha} f\right\|_{d/(d+1),\infty} = \sup_{\rho>0} \rho\lambda\left(\sigma_*^{q,\alpha} f > \rho\right)^{(d+1)/d} \le C \left\|f\right\|_{H_{d/(d+1)}}.$$

If $q = 2$ *and* $\alpha > (d-1)/2$ *then the same holds with the critical index* $d/(d/2+\alpha+1/2)$ *instead of* $d/(d+1)$.

Theorem 16 (Stein, Taibleson, Weiss, Oswald). *If $q = \infty$ and $\alpha = 1$ (resp. $q = 2$) then the operator $\sigma_*^{q,\alpha}$ is not bounded from $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ if p is smaller or equal to the critical index $d/(d+1)$ (resp. $d/(d/2+\alpha+1/2)$).*

Corollary 3 (Marcinkiewicz, Zhizhiashvili, Stein, Weiss, Oswald, Berens, Weisz). *Suppose that $q = 1, \infty$ and $\alpha > 0$ or $q = 2$ and $\alpha > (d-1)/2$. If $f \in L_1(\mathbb{T}^d)$ then*

$$\left\|\sigma_*^{q,\alpha} f\right\|_{1,\infty} = \sup_{\rho>0} \rho\lambda\left(\sigma_*^{q,\alpha} f > \rho\right) \le C \left\|f\right\|_1 .$$

Corollary 4 (Marcinkiewicz, Zhizhiashvili, Oswald, Stein, Weiss, Berens, Weisz). *Suppose that $q = 1, \infty$ and $\alpha > 0$ or $q = 2$ and $\alpha > (d-1)/2$. If $f \in L_1(\mathbb{T}^d)$ then*

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad a.e.$$

9. Circular Bochner-Riesz means $(q = \gamma = 2)$

Theorem 17 (Tao). *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq (d-1)/2$ and*

$$1 < p < \frac{2d-1}{d+2\alpha} \qquad \textit{or} \qquad p > \frac{2d}{d-1-2\alpha},$$

then the maximal operator $\sigma_*^{2,\alpha}$ is not bounded on $L_p(\mathbb{T}^d)$ (see Figure 6).

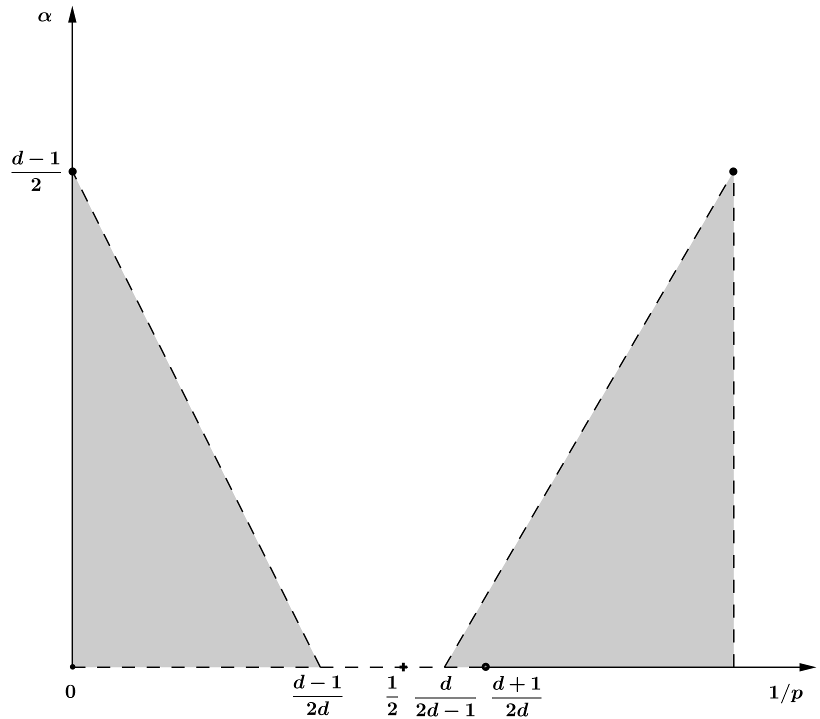

FIGURE 6. Unboundedness of $\sigma_*^{2,\alpha}$ on $L_p(\mathbb{T}^d)$.

Theorem 18 (Carbery)**.** *Suppose that $d = 2$ and $q = \gamma = 2$. If $0 < \alpha \leq 1/2$ and*

$$\frac{2}{1+2\alpha} < p < \frac{4}{1-2\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ is bounded on $L_p(\mathbb{T}^d)$ (see Figure 7).

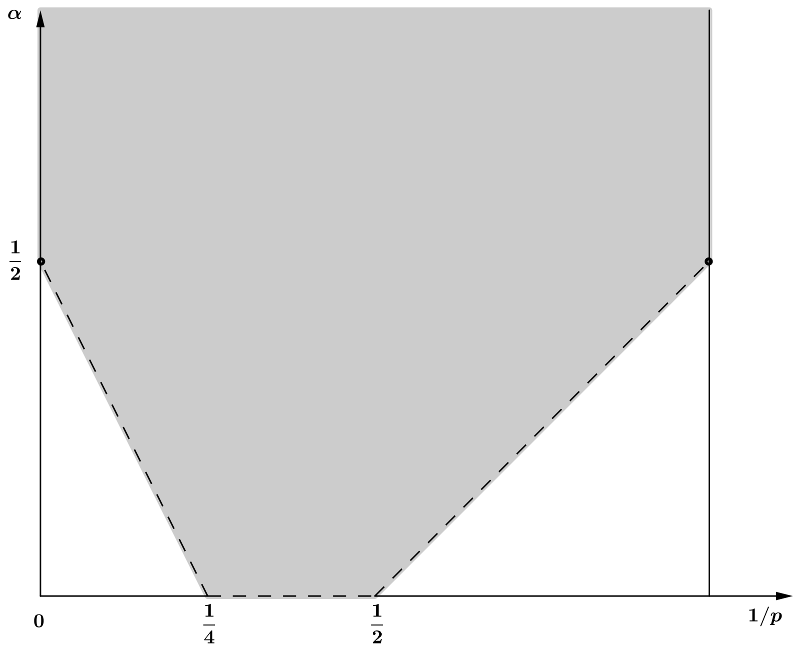

FIGURE 7. Boundedness of $\sigma_*^{2,\alpha}$ on $L_p(\mathbb{T}^d)$ when $d = 2$.

Theorem 19 (Christ). *Suppose that* $d \geq 3$ *and* $q = \gamma = 2$. *If* $\frac{d-1}{2(d+1)} \leq \alpha \leq \frac{d-1}{2}$ *and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ *is bounded on* $L_p(\mathbb{T}^d)$ *(see Figure 8).*

Theorem 20 (Stein and Weiss). *Suppose that* $d \geq 3$ *and* $q = \gamma = 2$. *If* $0 < \alpha < \frac{d-1}{2(d+1)}$ *and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2(d-1)}{d-1-4\alpha},$$

then the maximal Bochner-Riesz operator $\sigma_*^{2,\alpha}$ *is bounded on* $L_p(\mathbb{T}^d)$ *(see Figure 8).*

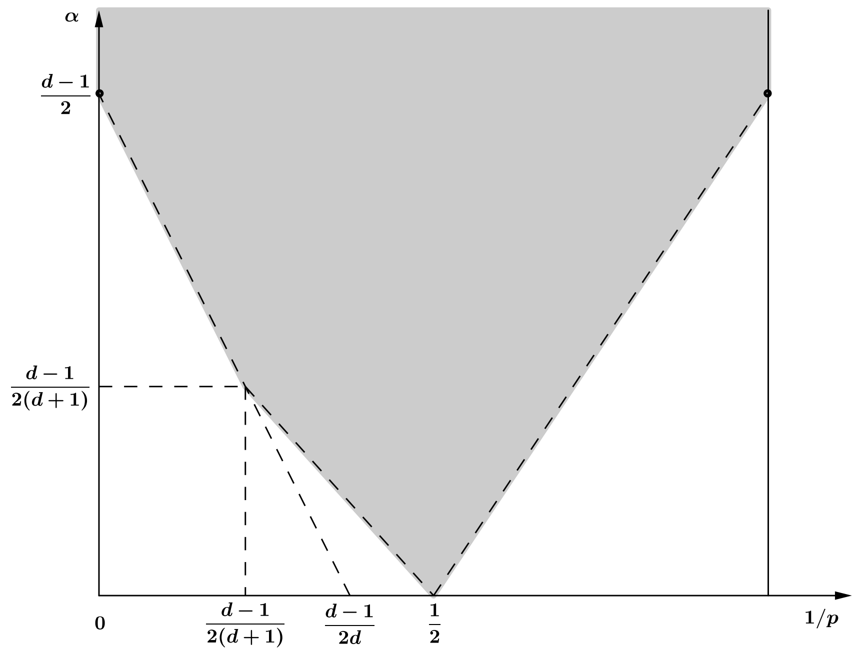

FIGURE 8. Boundedness of $\sigma_*^{2,\alpha}$ on $L_p(\mathbb{T}^d)$ when $d \geq 3$.

It is still an open question as to whether $\sigma_*^{2,\alpha}$ is bounded or unbounded in the region of Figure 9. If $d = 2$, then the question is open on the right hand side of the region of Figure 9 only, i.e., for $1/p \geq 1/2$.

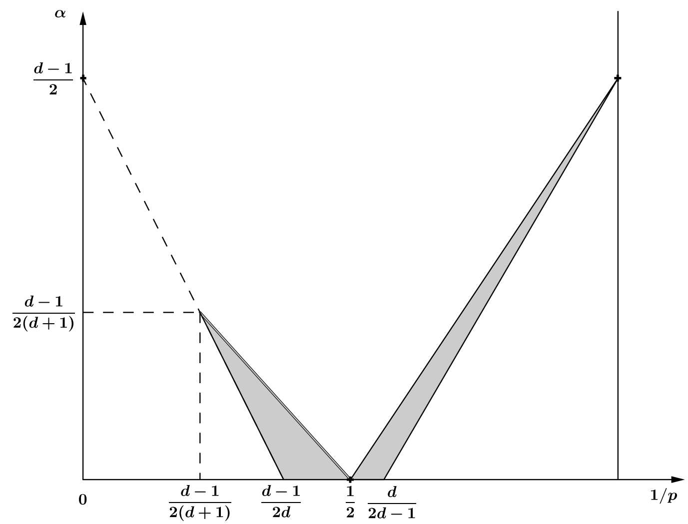

FIGURE 9. Open question of the boundedness of $\sigma_*^{2,\alpha}$ when $d \geq 3$.

Theorem 21 (Carbery, Rubio de Francia and Vega). *Suppose that $d \geq 2$ and $q = \gamma = 2$. If $0 < \alpha \leq (d-1)/2$ and*

$$\frac{2(d-1)}{d-1+2\alpha} < p < \frac{2d}{d-1-2\alpha},$$

then for all $f \in L_p(\mathbb{T}^d)$

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad a.e.$$

(see Figure 10).

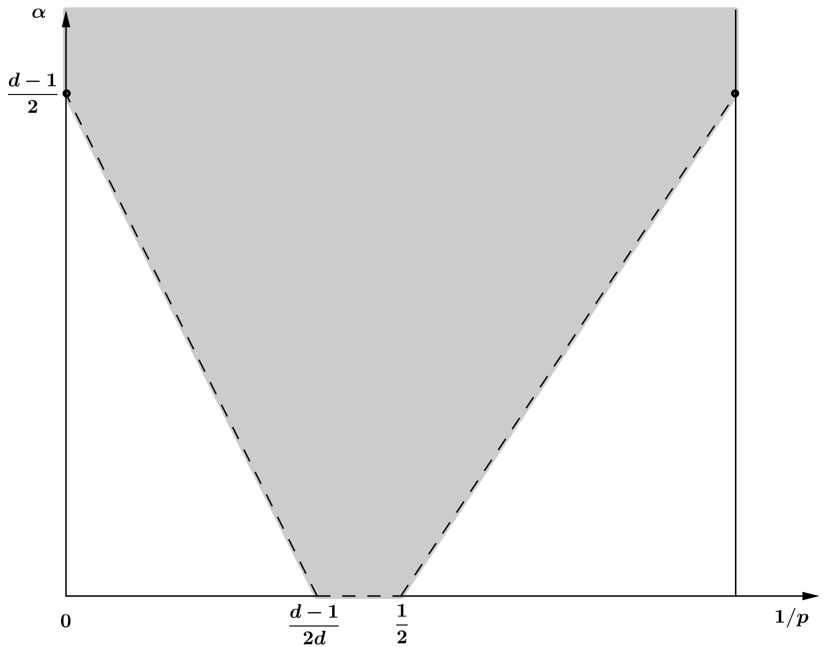

FIGURE 10. Almost everywhere convergence of $\sigma_n^{q,\alpha} f$, $f \in L_p(\mathbb{T}^d)$.

This talk is based on my work:

F. Weisz: Summability of Multi-Dimensional Trigonometric Fourier Series. Surveys in Approximation Theory, 7, 1-179, 2012