Spectral Analysis and Synthesis on Abelian Groups

4th Workshop on Fourier Analysis and Related Fields
26-30 August, 2013, Budapest, Hungary

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Notation and terminology

G: locally compact Abelian group, $\mathcal{C}(G)$: locally convex topological vector space of all continuous complex valued functions on G, topology: compact convergence

$\mathcal{M}_c(G)$: measure algebra $\approx$ the dual of $\mathcal{C}(G)$ $\approx$ linear space of all compactly supported measures G: commutative algebra with identity

Convolution:

$$\mu * \nu(f) = \int\int f(x+y)\, d\mu\, d\nu, \quad \mu * f(x) = \int f(x-y)\, d\mu(y)$$

Vector module: $\mathcal{C}(G)$ over $\mathcal{M}_c(G)$

Dirac–measure: $\delta_y(f) = f(y)$, δ_0 is the identity in $\mathbb{C}G$

Convolution operator: $f \mapsto \mu * f$

Translation: $\tau_y f = \delta_{-y} * f$

Variety, generated variety: $\tau(f)$; closed submodules are exactly the varieties

Notation and terminology

Special case:

G: discrete Abelian group, $\mathcal{C}(G)$: locally convex topological vector space of all complex valued functions on G, topology: pointwise convergence

$\mathbb{C}G$: group algebra $\approx$ the dual of $\mathcal{C}(G)$ $\approx$ linear space of all finitely supported complex measures on G $\approx$ linear space of all finitely supported complex functions on G: commutative algebra with

Convolution:

$$\mu * f(x) = \sum_y f(x-y)\,\mu(y)$$

Generators: $\delta_y(f) = f(y)$, δ_0 is the identity in $\mathbb{C}G$

Spectral analysis and synthesis

Spectral analysis for a variety: it has a nonzero finite dimensional subvariety

Synthesizable variety: all nonzero finite dimensional subvarieties span a dense subspace

Spectral synthesis for a variety: each subvariety is synthesisable

Spectral analysis on a group: spectral analysis holds for each nonzero variety

Spectral synthesis on a group: spectral synthesis holds for each variety

Spectral analysis: **there are** nonzero finite dimensional subvarieties

Spectral synthesis: **there are sufficiently many** nonzero finite dimensional subvarieties

History and results

Theorem

(Laurent Schwartz, 1948) Spectral synthesis holds on the reals.

Theorem

(Bernard Malgrange, 1954) For any nonzero linear partial differential operator $P(D)$ in $\mathbb{R}^n$ spectral synthesis holds for the solution space of the partial differential equation $P(D)f = 0$.

Theorem

(Leon Ehrenpreis, 1955) Spectral synthesis holds for each variety in $\mathcal{E}(\mathbb{C}^n)$ whose annihilator is a principal ideal.

Theorem

(Marcel Lefranc, 1958) Spectral synthesis holds on $\mathbb{Z}^n$.

History and results

Theorem

(Robert J. Elliott, 1965) Spectral synthesis holds on every discrete Abelian group.

Theorem

(Dmitrii I. Gurevič, 1975) Spectral synthesis fails to hold on $\mathbb{R}^n$*, if* $n \geqslant 2$.

Remark

(Zbigniew Gajda, 1987) Elliott's proof has several gaps.

Theorem

(L. Sz., 2004) Spectral synthesis fails to hold on the free Abelian group of countable generators.

History and results

Theorem

(L. Sz., 2004) Spectral synthesis fails to hold on any Abelian group with infinite torsion free rank.

Theorem

(Miklós Laczkovich and Gábor Székelyhidi, 2005) Spectral analysis holds on an Abelian group if and only if its torsion free rank is less than the continuum.

Theorem

(Miklós Laczkovich and L. Sz., 2007) Spectral synthesis holds on an Abelian group if and only if its torsion free rank is finite.

Annihilators

V: subset in $\mathcal{C}(G)$, $\quad V^\perp = \{\mu : \mu(f) = 0 \text{ for all } f \in V\}$

I: subset in $\mathbb{C}G$, $\quad I^\perp = \{f : \mu(f) = 0 \text{ for all } \mu \in I\}$

Theorem

$$(V^\perp)^\perp = V \text{ for each variety } V, \quad (I^\perp)^\perp = I \text{ for each ideal } I$$

Theorem

$(V_\gamma)_{\gamma\in\Gamma}$: a family of varieties, then $\left(\sum_\gamma V_\gamma\right)^\perp = \bigcap_\gamma V_\gamma^\perp$

Theorem

$(I_\gamma)_{\gamma\in\Gamma}$: a family of ideals, then $\left(\sum_\gamma I_\gamma\right)^\perp = \bigcap_\gamma I_\gamma^\perp$

Exponentials

Exponential: homomorphism of G into the (multiplicative) group of nonzero complex numbers:

$$m(x+y) = m(x)\,m(y), \;\; m(0) = 1$$

Theorem

Let G be an Abelian group and $f : G \to \mathbb{C}$ a function. Then the following conditions are equivalent.

1. *f is an exponential.*
2. *$\tau(f)$ is one dimensional and $f(0) = 1$.*
3. *$\tau(f)^{\perp}$ is the kernel of a multiplicative functional of $\mathbb{C}G$ and $f(0) = 1$.*
4. *$\mathbb{C}G/\tau(f)^{\perp} \cong \mathbb{C}$ and $f(0) = 1$.*

We call the maximal ideal M in the commutative ring R with unit *exponential*, if $R/M \cong \mathbb{C}$. Hence the exponential maximal ideals of $\mathbb{C}G$ are exactly the kernels of the multiplicative functionals, or, in other words, the annihilators of the exponentials.

Modified differences

Modified difference: $f : G \to \mathbb{C}$, y in G

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0$$

$$\Delta_{f;y_1,y_2,\dots,y_{n+1}} = \prod_{i=1}^{n+1} \left(\delta_{-y_i} - f(y_i)\,\delta_0\right)$$

Theorem

Let G be an Abelian group, $f : G \to \mathbb{C}$ a function. The ideal M_f generated by all modified differences $\Delta_{f;y}$ with y in G is proper if and only if f is an exponential and $M_f = \tau(f)^{\perp}$.

The function $f : G \to \mathbb{C}$ is called a *generalized exponential monomial*, if its annihilator contains some positive power of an exponential maximal ideal:

$$M_m^{n+1} \subseteq \tau(f)^{\perp}$$

If f is nonzero, then $\tau(f)^{\perp} \subseteq M_m$.

Characterization of generalized exponential monomials

If f is a nonzero generalized exponential monomial, then the maximal ideal M with $\tau(f)^{\perp} \subseteq M$ is unique. In particular, the exponential m with $\tau(f)^{\perp} \subseteq M_m$ is unique (degree). A generalized exponential monomial is called *exponential monomial*, if its variety is finite dimensional.

Theorem

The function $f : G \to \mathbb{C}$ is a nonzero generalized exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local ring with nilpotent exponential maximal ideal.

Theorem

The function $f : G \to \mathbb{C}$ is a nonzero exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local Artin ring.

Theorem

The function $f : G \to \mathbb{C}$ is a nonzero exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local Noether ring with nilpotent exponential maximal ideal.

Generalized exponential polynomials

Sums of generalized exponential monomials are called *generalized exponential polynomials*. Uniqueness, linear independence, etc.

$$f(x) = \varphi_1 + \varphi_2 + \cdots + \varphi_n$$

A generalized exponential polynomial is called an *exponential polynomial*, if its variety is finite dimensional. Equivalent: sum of exponential monomials.

Characterization of exponential polynomials

Theorem

The function $f : G \to \mathbb{C}$ is a generalized exponential polynomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is a semi-local ring with exponential maximal ideals and nilpotent Jacobson radical.

Theorem

The function $f : G \to \mathbb{C}$ is an exponential polynomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is an Artin ring.

Theorem

The function $f : G \to \mathbb{C}$ is an exponential polynomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is a semi-local Noether ring with exponential maximal ideals and nilpotent Jacobson radical.

Spectral analysis and synthesis

Theorem

Let G be an Abelian group and V a variety on G. Then the following statements are equivalent:

1. *Spectral analysis holds for V.*
2. *There is an exponential in V.*
3. *There is a nonzero exponential monomial in V.*
4. *There is a nonzero exponential polynomial in V.*

Theorem

Let G be an Abelian group and V a variety on G. Then the following statements are equivalent:

1. *V is synthesizable.*
2. *All exponential monomials in V span a dense subspace.*
3. *The exponential monomials in V form a dense subset.*

Spectral analysis and synthesis

Theorem

Let G be an Abelian group and V a variety on G. Spectral analysis holds for V if and only if its annihilator is included in an exponential maximal ideal.

Corollary

Spectral analysis holds on an Abelian group if and only if each maximal ideal of its group algebra is exponential.

Theorem

Let G be an Abelian group and V a variety on G. Then V is synthesizable if and only if $\mathbb{C}G/V^\perp$ can be embedded into a direct product of local Artin rings.

Corollary

Let G be an Abelian group. Spectral synthesis fails to hold for a variety, if it contains a generalized exponential monomial, which is not an exponential monomial.