On the almost everywhere convergence of two-dimensional Cesaro means on the 2-adic additive group

ILONA SIMON

University of Pécs, Hungary

GYÖRGY GÁT

College of Nyíregyháza, Hungary

4th Workshop on Fourier Analysis and related fields

2013, Budapest.

1923, Riesz: If $f \in L^1([0,1])$ and $\alpha > 0$, then $\sigma_n^\alpha f \to f$ a.e. $(n \to \infty)$.

In 1939, Marcinkiewicz and Zygmund proved that the Fejér means $\sigma_n^1 f$ of the trigonometric Fourier series of two variable integrable functions converge almost everywhere to the function if the ratio of the indices of the means remain in some positive cone around the identical function as they tend to infinity.

1935, Jessen, Marcinkiewicz and Zygmund for trigonometric Fourier series of two variable functions: If $f \in L \log^+ L$ $\sigma_{n,m}^1 f \to f$ a.e. $(\min n, m \to \infty)$.

Results with respect to the characters of the 2-adic additive group:

1997, Gát: If $f \in L^1$, then $\sigma_n^1 f \to f$ a.e. $(n \to \infty)$.

Gát: If $f \in L^1$ and $\alpha > 0$, then $\sigma_n^\alpha f \to f$ a.e. $(n \to \infty)$.

2. Basic notions and tools

Consider the unit interval $\mathbb{I} := [0, 1)$,

the cartesian products $\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}$ and $\mathbb{I}^2 := \mathbb{I} \times \mathbb{I}$.

Denote the 1- and 2-dimensional Haar measure of subsets $E \subseteq \mathbb{I}$ and $F \subseteq \mathbb{I}^2$ by $\mu(E) = |E|$ and $\mu_2(F) = |F|$.

Set $\mathcal{I} := \{[p/2^n, (p+1)/2^n) \mid p, n \in \mathbb{N}, 0 \leq p < 2^n\}$ the set of dyadic intervals.

Let $\mathcal{I}^2 := \left\{I^2 = I_1 \times I_2 \mid I_1, I_2 \in \mathcal{I},\ |I_1| = |I_2|\right\}$ denote the collection of dyadic squares.

The dyadic expansion of $x \in \mathbb{I}$ is

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)},$$

where $x_n \in \{0, 1\}$. If x is a dyadic rational, that is $x \in \{\frac{p}{2^n} : p, n \in \mathbb{N},\ 0 \leq p < 2^n\}$, we choose the expansion which terminates in 0 's.

The dyadic maximal function of an $f \in L^1(\mathbb{I}^j)$ is defined by

$$f^*(x) = \sup_m \frac{1}{|I_m(x)|}\left| \int_{I_m(x)} f \right| \qquad (x \in \mathbb{I}^j, j \in \{1,2\}). \tag{1}$$

Furthermore the dyadic Hardy space $H(\mathbb{I}^j) := \{f \in L^1(\mathbb{I}^j) \mid \|f\|_H := \|f^*\|_1 < \infty\}$.

The 2-adic sum of $a, b \in \mathbb{I}$ is $a + b := \sum_{n=0}^{\infty} s_n 2^{-(n+1)}$, where bits $q_n, s_n \in \{0,1\}$ $(n \in \mathbb{N})$ are defined recursively as follows: $q_{-1} := 0$, $a_n + b_n + q_{n-1} = 2q_n + s_n$ for $n \in \mathbb{N}$. The group $(\mathbb{I}, +)$ is called the group of 2-adic integers.

Set

$$v_{2^n}(x) := exp\left(2\pi\imath\left(\frac{x_n}{2} + \cdots \frac{x_0}{2^{n+1}}\right)\right) \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$

and

$$v_n := \prod_{i=0}^{\infty} v_{2^i}^{n_i},$$

where $n \in \mathbb{N}$ has dyadic expansion $n = \sum_{i=0}^{\infty} n_i 2^i$ $(n_i \in \{0,1\}(i \in \mathbb{N}))$.

Denote by

$$\hat{f}(n) := \int_{\mathbb{I}} f\overline{v}_n d\lambda, \qquad D_n := \sum_{k=0}^{n-1} v_k$$

the Fourier coefficients, and the Dirichlet kernels.

Denote by K_n^α the (C,α) kernel for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$K_n^\alpha := \frac{1}{A_n^\alpha}\sum_{i=0}^{n} A_{n-i}^{\alpha-1} D_i, \quad A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k).$$

The (C,α) Cesàro means of the integrable function f is

$$\sigma_n^\alpha f(y) := \frac{1}{A_n^\alpha}\sum_{i=0}^{n} A_{n-i}^{\alpha-1} S_i(y) = \int_{\mathbb{I}} f(x) K_n^\alpha(y-x) d\mu(x) = f * K_n^\alpha \quad (n \in \mathbb{N}, y \in \mathbb{I}).$$

Define the two-dimensional character system $(v_n, n \in \mathbb{N}^2)$, Dirichlet kernel functions and for $\alpha, \beta \in \mathbb{R}$ the Cesàro kernel functions on $\mathbb{I}^2$ as the Kronecker products of the one-dimensional functions:

$$v_n(x) := v_{n_1}(x_1)v_{n_2}(x_2), \quad D_n(x) := D_{n_1}(x_1)D_{n_2}(x_2)$$

$$K_n^{\alpha,\beta}(x) := K_{n_1}^{\alpha}(x_1)K_{n_2}^{\beta}(x_2) \quad (x = (x_1, x_2) \in \mathbb{I}^2, n = (n_1, n_2) \in \mathbb{N}^2).$$

Now, the two-dimensional Fourier coefficients, the n-th rectangular partial sum of the Fourier series and the n-th (C, α) means of $f \in L^1(\mathbb{I}^2)$ are

Note, that $\sigma_{n_1,n_2}^{\alpha,\beta} f(y) = f * (K_{n_1}^{\alpha} \times K_{n_2}^{\beta})$.

3. Cesaro summability

Theorem 1. *Let* $\alpha, \beta > 0$, $f \in L\log^+ L(\mathbb{I}^2)$. *Then we have a.e. convergence* $\sigma_{n,m}^{\alpha,\beta} f \to f$ *as* $m, n \to \infty$.

Consider the one-dimensional maximal operators

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|,$$

$$\sigma^\alpha f := \sup_{n\in\mathbb{N}} \int_{\mathbb{I}} \Big| f(u) K_n^\alpha(x-u) \Big| d\mu(u)$$

$$\widetilde{\sigma}^\alpha f := \sup_{n\in\mathbb{N}} \Big| \int_{\mathbb{I}} f(u) \Big| K_n^\alpha(x-u) \Big| d\mu(u) \Big|.$$

Lemma 2 (Gát)**.** *The operator* $\sigma_*^\alpha f$ *is of type* (H, L), *of type* (L^p, L^p) $(p > 1)$ *and of weak type* (L^1, L^1), *furthermore* $\|K_n^\alpha\|_1 \le C_\alpha$.

Lemma 3 (Gát, I. Simon)**.** *The operator* $\widetilde{\sigma}^\alpha f$ *is of weak type* (L^1, L^1).

Lemma 4 (Gát, I. Simon)**.** *The operator* $\sigma^\alpha f$ *is of weak type* (L^1, L^1).

Consider the two-dimensional maximal operator $\sigma_*^{\alpha,\beta} f := \sup_{n,m\in\mathbb{N}} |\sigma_{n,m}^{\alpha,\beta} f|$.

Lemma 5 (Gát, I. Simon). *There is a constant $C > 0$ such that*

(2) $$\mu_2(\{(x,y) \in \mathbb{I}^2 : \sigma_*^{\alpha,\beta} f > \lambda\}) \leq \frac{C \cdot C_\beta}{\lambda} \|f\|_\sharp \quad (f \in H^\sharp(\mathbb{I}^2),\ \lambda > 0).$$

Proof: Let us consider $f_u : \mathbb{I} \to \mathbb{C},\ f_u(v) := f(u,v)$ and

$$g_y(u) := \sup_m \left| \int_{\mathbb{I}} f(u,v) K_m^\beta(y-v) d\mu(v) \right| \qquad (u \in \mathbb{I})$$

that is, $g_y(u) = (\sigma_*^\beta f_u)(y)$. Now,

$$|(\sigma_{n,m}^{\alpha,\beta} f)(x,y)| = \left| \int_{\mathbb{I}^2} f(u,v) K_n^\alpha(x-u) K_m^\beta(y-v) d\mu(v) d\mu(u) \right| \leq$$

$$\leq \sup_n \int_{\mathbb{I}} \left| K_n^\alpha(x-u) \right| \left(\sup_m \left| \int_{\mathbb{I}} f(u,v) K_m^\beta(y-v) d\mu(v) \right| \right) d\mu(u) =$$

$$= \sup_n \int_{\mathbb{I}} \left| K_n^\alpha(x-u) \right| g_y(u) d\mu(u) = (\sigma^\alpha g_y)(x).$$

Thus

(3) $$(\sigma_*^{\alpha,\beta} f)(x,y) \leq (\sigma^\alpha g_y)(x) \quad (x, y \in \mathbb{I}).$$

Let us use the one-dimensional Haar measure

$$m(y) := \mu(\{x \in \mathbb{I} : (\sigma_*^{\alpha,\beta} f)(x,y) > \lambda\}).$$

Now, (3) and Lemma 2 implies:

$$m(y) \le \mu(\{x \in \mathbb{I} : (\sigma^{\alpha} g_y)(x) > \lambda\}) \le \frac{C}{\lambda}\|g_y\|_1.$$

Thus,

$$\mu_2(\{(x,y) \in \mathbb{I}^2 : (\sigma_*^{\alpha,\beta} f)(x,y) > \lambda\}) = \int_{\mathbb{I}} m(y) d\mu(y) \le \frac{C}{\lambda}\int_{\mathbb{I}} \|g_y\|_1 d\mu(y) =$$

$$= \frac{C}{\lambda}\int_{\mathbb{I}}\int_{\mathbb{I}} g_y(x) d\mu(y) d\mu(x) = I_1.$$

Now, by Lemma 1 we have

$$\int_{\mathbb{I}} g_y(x) d\mu(y) = \int_{\mathbb{I}} (\sigma_*^{\beta} f_x)(y) d\mu(y) = \|\sigma_*^{\beta} f_x\|_1 \le C_\beta \|f_x\|_H = C_\beta \|f^{\sharp}\|_1 = C_\beta \|f\|_{\sharp}.$$

□

Proof of Theorem 1: A usual density argument and Lemma 3 implies the proof.

[1] Gát, Gy., *Almost everywhere convergence of Cesàro means of Fourier series on the group of 2-adic integers*, Acta Math. Hungar., 116 (3) (2007), pp. 209-221.

[2] Gát, Gy., *On the almost everywhere convergence of Fejér means of functions on the group of 2-adic integers*, J. Approx. Theory, 90 (1997), pp. 8896.

[3] Móricz, F., Schipp, F., Wade, W., R., *Cesàro summability of double Walsh-Fourier series* Trans. Amer. Math. Soc., 329 (1992), pp. 131-140.

[4] Schipp, F., Wade, W.R., Simon, P., Pál, J., *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Ltd., Bristol and New York, (1990).

[5] Riesz, M. *Sur la sommation des s´eries de Fourier. Acta Sci. Math.* (Szeged), 1, pp. 104-113, 1923.

[6] Jessen, B., Marcinkiewicz, J., Zygmund, A. . *Note on the differentiability of multiple integrals.* Fund. Math., 25 (1935), pp. 217-234.