A FAMILY OF SMOOTH COMPACTLY SUPPORTED TIGHT WAVELET FRAMES

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Definition

A sequence $\{\phi_n\}_{n=1}^{\infty}$ *of elements in a separable Hilbert space* $\mathbb{H}$ *is a* ***frame*** *for* $\mathbb{H}$ *if there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \phi_n\rangle|^2 \leq C_2\|h\|^2, \qquad \forall h \in \mathbb{H},$$

where $\langle \cdot, \cdot \rangle$ *denotes the inner product on* $\mathbb{H}$.

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where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{H}$.

The constants C_1 and C_2 are called *frame bounds*.

The definition implies that a frame is a complete sequence of elements of $\mathbb{H}$.

A frame $\{\phi_n\}_{n=1}^{\infty}$ is tight if we may choose $C_1 = C_2$.

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A set of functions $\Psi = \{\psi_1, \dots, \psi_N\} \subset L^2(\mathbb{R}^d)$ is called a wavelet frame or framelet with dilation A, if the system

$$\{|\det A|^{j/2}\psi_\ell(A^j\mathbf{x} + \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \leq \ell \leq N\} \tag{1}$$

is a frame for $L^2(\mathbb{R}^d)$.

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If the system (1) is an orthonormal basis for $L^2(\mathbb{R}^d)$ then Ψ is called a wavelet.

It follows from the definition of tight wavelet frame that any function $f \in L^2(\mathbb{R}^d)$ has the "wavelet" expansion

$$f(\mathbf{x}) = \sum_{\ell=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{1}{C_1} \langle f, \psi_\ell^{(j,\mathbf{k})} \rangle \, \psi_\ell^{(j,\mathbf{k})}(\mathbf{x}).$$

where $\psi_\ell^{(j,\mathbf{k})}(\mathbf{x}) = |\det A|^{j/2} \psi_\ell(A^j\mathbf{x} + \mathbf{k})$.

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When A is a 2×2 dilation matrix with integer entries and $|\det A| = 2$, we show a method to construct a family of tight wavelet frames with only three smooth compactly supported generators.

Our construction is made in the Fourier transform side.

The Fourier transform of $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\mathbf{y}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{y}}\, d\mathbf{x}.$$

Theorem ((UEP), Han, Ron and Shen, (1997))

Let $\phi \in L^2(\mathbb{R}^d)$ *be compactly supported such that* $\widehat{\phi}(A^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t})$, *where* P *is a trigonometric polynomial, and* $|\widehat{\phi}(\mathbf{0})| = 1$. *Assume there are trigonometric polynomials or rational functions* Q_ℓ, $\ell = 1, \cdots, N$, *that satisfy the UEP condition*

$$P(\mathbf{t})\overline{P(\mathbf{t}+\mathbf{j})} + \sum_{\ell=1}^{N} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} \tag{2}$$

$$= \begin{cases} 1 & \text{if} \quad \mathbf{j} \in \mathbb{Z}^d, \\ 0 \quad \text{if} \quad \mathbf{j} \in & ((A^*)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d) \setminus \mathbb{Z}^d \end{cases}$$

If

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \ldots, N,$$

then $\Psi = \{\psi_1, \ldots, \psi_N\}$ *is a tight framelet in* $L^2(\mathbb{R}^d)$ *with dilation matrix* A *and frame bound 1.*

- Daubechies (1988): on $\mathbb{R}$, $A = 2$
- Ron and Shen (1998): on $\mathbb{R}^2$, $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
- Gröchenig and Ron (1998): $\mathbb{R}^d$, any A.
- Han (2003): on $\mathbb{R}^d$, any A.

Lai and Stöckler (2006) for $A = 2$ and $d = 1$.

Theorem

Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map preserving the integer lattice and let $\boldsymbol{\Gamma}_{A^*} = \{\mathbf{p}_s\}_{s=0}^{d_A-1}$ be a full collection of representatives of the cosets of $(A^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$. Let $P(\mathbf{t})$ be a trigonometric polynomial defined on $\mathbb{R}^d$ such that $\sum_{s=0}^{d_A-1} |P(\mathbf{t}+\mathbf{p}_s)|^2 \leq 1$. Suppose that there exist trigonometric polynomials $\widetilde{P}_1(A^*\mathbf{t}), \ldots, \widetilde{P}_M(A^*\mathbf{t})$ such that

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t}+\mathbf{p}_s)|^2 + \sum_{j=1}^{M} |\widetilde{P}_j(A^*\mathbf{t})|^2 = 1. \qquad (3)$$

Then the there exist trigonometric polynomials P and Q_ℓ, $\ell = 1, \ldots, |\det A| + M$, satisfy the identity (2).

Our construction:

For a given $n \in \mathbb{N}$, denote

$$H(\mathbf{t}) := \frac{1}{d_A} \sum_{s=0}^{d_A-1} e^{-2\pi i \mathbf{t}\cdot\mathbf{q}_s} \qquad \text{and} \qquad P(\mathbf{t}) = |H(\mathbf{t})|^{2n}, \tag{4}$$

where $\{\mathbf{q}_s\}_{s=0}^{d_A-1}$ is a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$.

Then

$$P(\mathbf{0}) = 1 \qquad \text{and} \qquad \sum_{s=0}^{d_A-1} |P(\mathbf{t}+\mathbf{p}_s)|^2 \leq 1.$$

The infinity product

$$\prod_{j=1}^{\infty} P((A^*)^{-j}\mathbf{t})$$

converges to a non negative continuous function $\widehat{\phi}$ in $L^2(\mathbb{R}^d)$ such that $\|\widehat{\phi}\|_{L^2(\mathbb{R}^d)} \leq 1$, $\widehat{\phi}(\mathbf{0}) = 1$ and it satisfies the refinement equation

$$\widehat{\phi}(A^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^d. \tag{5}$$

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Then $\phi \in L^2(\mathbb{R}^d)$ such that its Fourier transform is $\widehat{\phi}$ is compactly supported.

Then there are numbers $\alpha_{\mathbf{k}}$ such that

$$1 - \sum_{s=0}^{d_A - 1} |P(\mathbf{t} + \mathbf{p}_s)|^2 = \sum_{\mathbf{k} \in \mathbb{Z}^d} \alpha_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot A^*(\mathbf{t})}, \quad \alpha_{\mathbf{k}} \in \mathbb{R}. \qquad (6)$$

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If Γ denotes the set of nonzero α_k, let the trigonometric polynomials $\widetilde{P}_{\mathbf{k}}$ be defined by

$$\widetilde{P}_{\mathbf{0}}(\mathbf{t}) = 0, \qquad \widetilde{P}_{\mathbf{k}}(\mathbf{t}) := \sqrt{\frac{|\alpha_{\mathbf{k}}|}{2}}(1 - e^{2\pi i \mathbf{k}\cdot\mathbf{t}}), \qquad \text{if } \mathbf{k} \in \Gamma \setminus \{\mathbf{0}\}, \tag{7}$$

Then

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t} + \mathbf{p}_s)|^2 + \sum_{\mathbf{k}\in\Gamma} |\widetilde{P}_{\mathbf{k}}(A^*\mathbf{t})|^2 = 1.$$

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Then

$$\sum_{s=0}^{d_A-1} |P(\mathbf{t} + \mathbf{p}_s)|^2 + \sum_{\mathbf{k}\in\Gamma} |\widetilde{P}_{\mathbf{k}}(A^*\mathbf{t})|^2 = 1.$$

Proof. Using the elementary formula $1 - \cos(2\pi\mathbf{k}\cdot\mathbf{t}) = \frac{1}{2}|1 - e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}|^2$.

By a theorem by Lai and Stöckler and by the UEP,

$$\widehat{\psi}_\ell(A^*\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \ldots, N.$$

Theorem

$\Psi = \{\psi_1, \ldots, \psi_N\}$ is a tight framelet in $L^2(\mathbb{R}^d)$ with dilation matrix A, and the functions ψ_ℓ are compactly supported.

The degree of smoothness “increases” with n.

OUR CONSTRUCTION IN $\mathbb{R}^2$

Let A be an 2×2 dilation matrix with integer entries such that $|\det A| = 2$.

Two matrices A and B with integer coefficients are *integrally similar* if there exists a matrix U with integer entries such that $|\det U| = 1$ and $A = U^{-1}BU$.

Lemma (Lagarias and Wang, 1995)

Let A be an 2×2 dilation matrix with integer entries such that $|\det A| = 2$. If $\det A = -2$ then A is integrally similar to A_1. If $\det A = 2$ then A is integrally similar to one of the matrices A_k, $k = 2, \ldots, 6$.

$$A_1 := \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},\ A_2 := \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix},\ A_3 := \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix},$$
$$A_4 := \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix},\ A_5 := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ A_6 := \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

We now focus on the dilation matrices A_k.
We have

$$\{\mathbf{p}_0, \mathbf{p}_1\} = \{(0,0)^T, (1/2,0)^T\}, \qquad k = 1,2,3,4$$

$$\{\mathbf{p}_0, \mathbf{p}_1\} = \{(0,0)^T, (1/2,1/2)^T\}, \qquad k = 5,6,$$

is a full collection of representatives of the cosets of $(A_k)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$.

Consider a matrix A_k, $k = 1, \ldots, 6$.

Let $m, n \in \mathbb{N}$, $\mathbf{t} = (t_1, t_2)$ and $P(\mathbf{t}) := \cos^{2n}(2m-1)\pi t_1$.

Consider a matrix A_k, $k = 1, \ldots, 6$.

Let $m, n \in \mathbb{N}$, $\mathbf{t} = (t_1, t_2)$ and $P(\mathbf{t}) := \cos^{2n}(2m-1)\pi t_1$.

Then $\phi \in L^2(\mathbb{R}^2)$, defined by

$$\widehat{\phi}(\mathbf{t}) = \prod_{j=1}^{\infty} P((A_k^*)^{-j}\mathbf{t}),$$

is non null and compactly supported.

Obviously,

$$\widehat{\phi}(A_k^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^2. \tag{8}$$

We have

$$|P(\mathbf{t})|^2+|P(\mathbf{t}+\mathbf{p}_1)|^2 = \cos^{4n}((2m-1)\pi t_1)+\sin^{4n}((2m-1)\pi(t_1)) \leq 1$$

Since the values of the trigonometric polynomial P only depend on one variable, from a lemma of Riesz we know there is a non null trigonometric polynomial $L(A_k^*\mathbf{t})$ on $\mathbb{R}^2$ such that

$$|L(A_k^*\mathbf{t})|^2 = 1 - \left(|P(\mathbf{t})|^2 + |P(\mathbf{t}+\mathbf{p}_1)|^2\right)$$

The coefficients of $L(A_k^*\mathbf{t})$ may be obtained by *spectral factorization*.

Let

$$Q_1(\mathbf{t}) := \frac{1}{\sqrt{2}}[1 - \cos^{4n}((2m-1)\pi t_1) - \cos^{2n}((2m-1)\pi t_1)\ \sin^{2n}((2m-1)\pi t_1)],$$

$$Q_2(\mathbf{t}) := \frac{e^{i2\pi t_1}}{\sqrt{2}}[1 - \cos^{4n}((2m-1)\pi t_1) + \cos^{2n}((2m-1)\pi t_1)\ \sin^{2n}((2m-1)\pi t_1)],$$

and

$$Q_3(\mathbf{t}) := -\frac{1}{2}\cos^{2n}((2m-1)\pi t_1)\ \overline{L(A_k^*\mathbf{t})},$$

Theorem

Let

$$\widehat{\psi_\ell}(A_k^*\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, 2, 3$$

Then $\Psi = \{\psi_1, \psi_2, \psi_3\}$ *is a tight framelet with dilation factor* A_k *and frame constant 1, and the functions* ψ_ℓ *have compact support.*

The degree of smoothness "increases" with n.

Corollary

Let A *dilation matrix preserving the integer lattice with* $|\det A| = 2$ *and let* $k \in \{1, \dots, 6\}$ *be such that there exists an integer matrix* U *with* $\det(U) = 1$*, such that* $A = U^{-1}A_kU$*.*
If

$$\theta_\ell(\mathbf{t}) = \psi_\ell(U\mathbf{t}) \qquad \ell = 1, 2, 3,$$

then $\Theta = \{\theta_1, \theta_2, \theta_3\} \subset L^2(\mathbb{R}^2)$ *is a tight framelet with* A*, and the functions* θ_ℓ *have compact support.*