LATTICE TILINGS OF $\mathbb{Z}^d$ BY TRANSLATED INTEGER SUBLATTICES

MACIEJ BORODZIK, DANNY NGUYEN, AND SINAI ROBINS

Abstract. When we represent $\mathbb{Z}^d$ as a finite disjoint union of translated integer sublattices, the translated sublattices must possess some special properties. We call such a representation a lattice tiling. We develop a theoretical framework, based on multiple residues and dual groups. We prove that in any such lattice tiling of $\mathbb{Z}^d$, if p is a prime and p^k divides the determinant of one translated sublattice, then p^k also divides the determinant of (at least) one of the other translated sublattices. We also investigate the question of when a lattice tiling must possess at least two translated sublattices which are translates of one another, and we give in general dimension a sufficient condition in terms of cyclic lattices.

1. Introduction

1.1. **Overview.** Suppose we decompose the integer lattice $\mathbb{Z}^d$ into a finite, disjoint union of integer translates of sublattices of $\mathbb{Z}^d$. We call such a decomposition of the integer lattice a *lattice tiling*. Given a lattice tiling, what can be said about the structure of the translated sublattices? For $d = 1$, an interesting question was posed by Erdős, namely whether there are always at least two arithmetic progressions (1-dimensional translated sublattices) which are translates of each other. Newman and Mirsky gave a classic but unpublished proof of the affirmative answer to this question, which combined combinatorial number theory with an analysis of generating functions, and was given almost immediately after it was posed by Erdős.

Extending these notions to higher dimensions, we say that a lattice tiling has the *translation property* if at least two of its translated sublattices are integer translates of each other. To motivate the results of this paper, we focus on the following three questions:

Question 1.1. What are some natural and general *necessary* conditions for the existence of a lattice tiling?

Question 1.2. In any general dimension d, are there some nice sufficient conditions for a lattice tiling to have the translation property?

Question 1.3. For $d = 2$, is there a lattice tiling which does not have the translation property?

In the process of trying to answer these questions, we develop some analytic tools which may be of independent interest. These tools involve generating functions associated to sublattices of $\mathbb{Z}^d$, residue calculus of holomorphic functions of several variables [12], and some elementary considerations. In the recent paper [6], the translation property for higher dimensional lattice tilings was considered from a discrete Fourier perspective. The translation

2010 *Mathematics Subject Classification.* primary: 52C10, secondary: 52C15, 52C17, 32A27.

Key words and phrases. lattice, sublattice, tiling, generating function, Pontryagin dual group, disjoint covering system.

The first author was supported by Polish OPUS grant No 2012/05/B/ST1/03195.

The second and third authors were supported in part by the Singapore MOE Tier 2 research grant MOE2011-T2-1-090.

property for dimension $d > 1$ was apparently first considered in another unpublished manuscript, this time an MIT Master's thesis by A. Schwartz [11], using purely combinatorial methods.

Although Question 1.3 remains open, we show that any potential counterexample to it must be large in the sense that there must be a lattice whose determinant is divisible by the squares of at least two primes. Using a brute force computer program we have shown that there is no counterexample if all the lattices have determinants bounded from above by 36.

On the other hand, for any dimension $d \geq 3$, there are counterexamples in [6] that show that there are 4 lattice translates which form a lattice tiling of $\mathbb{Z}^d$, but which do not have the translation property. We discuss these examples in Section 7 and classify all possible translation-free tilings that consist of at most four lattices.

1.2. **Terminology and notation.** We now provide concise definitions of notions that will be used throughout the paper.

We call $\mathcal{L}$ a *lattice* if it is a finite index subgroup of $\mathbb{Z}^d$, of full rank. The index of $\mathcal{L}$ in $\mathbb{Z}^d$ is called the *determinant* of $\mathcal{L}$. Throughout the article, we write $d-$dimensional vectors in bold, to distinguish them from scalars. Thus any vector $\mathbf{v}$ has coordinates $(v_1, \dots, v_d)$, and we furthermore write $\mathbf{v} \geq 0$ if $v_1, \dots, v_d \geq 0$. We write $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := (1, \dots, 1)$.

Suppose now that $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_d) \in \mathbb{C}^d$ and $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d) \in \mathbb{Z}^d$. We define

$$\chi_{\mathbf{z}}(\mathbf{v}) = \mathbf{z}_1^{\mathbf{v}_1} \dots \mathbf{z}_d^{\mathbf{v}_d} \in \mathbb{C}.$$

Although it is not technically traditional, and such a homomorphism is sometimes called a quasi-character in algebraic number theory [7], we shall refer to $\chi_{\mathbf{z}}$ simply as a *character*. For a lattice $\mathcal{L} \subset \mathbb{Z}^d$, we call a complex point $\mathbf{z} \in \mathbb{C}^d$ a *dual point* of $\mathcal{L}$ if $\chi_{\mathbf{z}}(\mathbf{v}) = 1$ for all $\mathbf{v} \in \mathcal{L}$. We write

$$\mathbb{T}^d = \{(z_1, \dots, z_d) \in \mathbb{C}^d \colon |z_1| = \dots = |z_d| = 1\}.$$

Each element $\mathbf{z} \in \mathbb{T}^d \subset \mathbb{C}^d$ gives us a character (in the standard sense) over $\mathbb{Z}^d$; that is, it provides a homomorphism from $\mathbb{Z}^d$ to S^1 given by $\mathbf{v} \mapsto \chi_{\mathbf{z}}(\mathbf{v})$. Since a point $\mathbf{z} \in \mathbb{T}^d$ is a dual point of $\mathcal{L}$ if and only if $\chi_{\mathbf{z}}$ restricted to $\mathcal{L}$ is trivial, we see that the dual points can be regarded as characters on the finite abelian group

$$G_{\mathcal{L}} := \mathbb{Z}^d / \mathcal{L},$$

which we call the *group of the lattice*. We note that all the coordinates of a dual point have modulus 1. We also say that a character has finite order if each z_j is additionally assumed to be a root of unity, a condition tantamount to saying that the image of $\rho_{\mathbf{z}}$ is finite. It is a standard fact that the dual points form a group, known as the Pontryagin dual to $G_{\mathcal{L}}$, and it is particularly useful that this group is isomorphic to $G_{\mathcal{L}}$. We call this group $\widehat{G}_{\mathcal{L}}$. We clearly have

$$|\widehat{G}_{\mathcal{L}}| = |G_{\mathcal{L}}| = \det \mathcal{L}.$$

For any integer vector $\mathbf{v} \in \mathbb{Z}^d$, we call the discrete set of vectors

$$\mathbf{v} + \mathcal{L} := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in \mathcal{L}\}$$

a *lattice translate* of $\mathcal{L}$. The vector $\mathbf{v}$ will be referred to as a *translate vector*. Thus, a more formal description of a lattice tiling is the existence of a collection of ($d-$dimensional) lattices $\mathcal{L}_1, \dots, \mathcal{L}_n$ and translate vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^d$ such that

$$\bigcup_{j=1}^{n} \{\mathbf{v}_j + \mathcal{L}_j\} = \mathbb{Z}^d,$$

and such that $\{\mathbf{v}_i + \mathcal{L}_i\} \cap \{\mathbf{v}_j + \mathcal{L}_j\} = \emptyset$ for all $i \neq j$. In other words, for any $\mathbf{w} \in \mathbb{Z}^d$ there exists a unique $j \in \{1, \dots, n\}$ such that $\mathbf{w} - \mathbf{v}_j \in \mathcal{L}_j$.

1.3. **Statement of results.** We first remark that if $\mathbf{v}_1 + \mathcal{L}_1, \dots, \mathbf{v}_n + \mathcal{L}_n$ are the translated sublattices of any lattice tiling, then for all i, j we must have

$$\gcd(\det \mathcal{L}_i, \det \mathcal{L}_j) > 1. \tag{1.4}$$

Although this particular result may be thought of as a warm-up, we give a formal proof of (1.4) in Proposition 2.5, in section 2.1. In dimension 1, this was most likely already known by Mirsky and Newman and is an elementary fact. One of our main results is the following necessary arithmetic condition on primes dividing the determinants of the lattice translates in any lattice tiling.

Theorem 1.5. *If we have a lattice tiling, and if there is a prime p such that p^k divides the determinant of one of the lattice translates, then p^k divides the determinant of another lattice translate.*

We give the proof of Theorem 1.5 in Section 6.3. We call a lattice $\mathcal{L}$ a *cyclic* lattice if the dual group $\widehat{G}_{\mathcal{L}}$ is a cyclic group. To give an answer to question (1.2), we have the following result, stated in terms of cyclic lattices.

Theorem 1.6. *If we have a lattice tiling such that the lattice translate with largest determinant is cyclic, then our lattice tiling has the translation property.*

In dimension 2 we can prove a stronger condition, made precise in Theorem 6.4. Finally, the following necessary and sufficient condition for a lattice tiling to exist, which we call the character formula, is a basic tool that we use often in our proofs.

Theorem 1.7. *Let $\chi_{\mathbf{z}}$ be any character of finite order then we have a lattice tiling with the lattice translates $\mathbf{v}_1 + \mathcal{L}_1, \dots, \mathbf{v}_n + \mathcal{L}_n$ if and only if*

$$\sum_{j:\, \mathbf{z} \in \widehat{G}_{\mathcal{L}_j}} \frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det \mathcal{L}_j} = \begin{cases} 1, & \textit{if } \mathbf{z} = (1, \dots, 1) \\ 0, & \textit{otherwise.} \end{cases} \tag{1.8}$$

The 'only if' part of Theorem 1.7 is stated and proved in Proposition 5.10. The 'if' part is Theorem 5.16.

We also address the problem of classifying tilings without the translation property, which we call translation-free lattice tilings, and Lemma 7.1, Proposition 7.2, and Proposition 7.3 in section 7 together give a complete classification of all lattice tilings that consist of at most four lattice translates.

1.4. **Historical overview.** Note that equation (1.8) is a basic relation between roots of unity, with rational coefficients. Indeed, the 1-dimensional case has been extensively studied using vanishing sums of roots of unity over the rationals. In dimension 1, a lattice tiling is also known in the literature as a Disjoint Covering System (DCS). Paul Erdős initiated the study of covering systems in general (which means that the arithmetic progressions may not necessarily be disjoint, see [5]), and Erdős credits the beautiful proof of the translation property that we have defined above, for dimension 1, to an unpublished paper by Mirsky and Newman, and independently to an unpublished paper of Davenport and Rado. Many interesting papers have since been written about the 1-dimensional case of lattice tilings, and for more background, including some fascinating results on vanishing sums of roots of unity, the reader may refer to [2], [3], [4], [9], [10], [13].

There was a related question in the context of a general finite nilpotent group G. Suppose that such a group G is partitioned into some cosets of some of its subgroups. Then the conjecture, known as the Herzog-Schonheim conjecture, was that at least two of the cosets must have the same index. The Herzog-Schonheim conjecture was solved in 1986, in the paper [1].

Since a lattice tiling may also be thought of as a covering of the group $\mathbb{Z}^d$ by a finite disjoint union of cosets of subgroups of $\mathbb{Z}^d$, it then follows from [1], in our abelian group setting, that any lattice tiling must contain at least two lattice translates of the same determinant. In other words, there are two lattice translates that must have the same volume for their fundamental domain. But almost any other question about these fundamental domains remains open.

Finally, we mention that Paul Erdős himself has been quoted as saying in 1995 that "Perhaps my favorite problem of all concerns covering systems".

Acknowledgements. We thank Professor Krzysztof Przeslawski for bringing the Master's thesis [11] to our attention. The project was started when M.B. and S.R. were visiting Renyi Institute in Budapest. We would like to thank the Renyi Institute for their hospitality. The authors S.R. and D.N. would like to thank the University of Warsaw for their hospitality, where this project was continued.

2. Lattices, generating functions, and characters

2.1. **The tiling condition.**

Definition 2.1. A tiling $\mathbf{v}_1 + \mathcal{L}_1, \dots, \mathbf{v}_n + \mathcal{L}_n$ of $\mathbb{Z}^d$ *splits* if $\{1, \dots, n\}$ can be partitioned into $I_1 \cup \dots \cup I_k$ with $k > 1$, $|I_j| > 0$ for $j = 1, \dots, k$, $|I_1| > 1$ so that for any $j = 1, \dots, k$ the union

$$\bigcup_{i \in I_j} \{\mathbf{v}_i + \mathcal{L}_i\}$$

is another lattice translate. Otherwise a tiling is called *primitive*. This definition merely captures the intuition that one might like to generate more complex lattice tilings by starting with a simple one, and splitting one of the existing lattice translates into new "coarser" lattice translates.

Lemma 2.2. *Assume that $\mathcal{L} \subset Z^d$ is a lattice with $\det \mathcal{L} = p$, for p a prime. Fix any integer vector $\mathbf{v} \notin \mathcal{L}$. Then we have $\operatorname{span}\{\mathcal{L}, \mathbf{v}\} = \mathbb{Z}^d$.*

Proof. Let $\mathcal{T}$ be the lattice generated by $\mathcal{L}$ and $\mathbf{v}$. Since $\mathbf{v} \notin \mathcal{L}$, we have $\mathcal{L} \subsetneq \mathcal{T} \subset \mathbb{Z}^d$ and the index of $\mathcal{T}$ in $\mathbb{Z}^d$ is therefore a proper divisor of p. We conclude that $\det \mathcal{T} = 1$ and hence $\mathcal{T} = \mathbb{Z}^d$. □

Proposition 2.3. *If we have $\det \mathcal{L}_k = p$ with p a prime for some k then the tiling either splits or all the lattices in the tiling are equal to $\mathcal{L}_k$.*

Proof. Assume that $\det \mathcal{L}_1 = p$ and translate the tiling if necessaray so that $\mathbf{v}_1 = 0$. By this assumption, we have $\mathcal{L}_1 \cap (\mathbf{v}_i + \mathcal{L}_i) = \emptyset$ for all $i > 1$, so that $\mathbf{v}_i \notin \operatorname{span}\{\mathcal{L}_1, \mathcal{L}_i\}$ and hence $\operatorname{span}\{\mathcal{L}_1, \mathcal{L}_i\} \subsetneq \mathbb{Z}^d$. From Lemma 2.2, we conclude that $\mathcal{L}_i \subset \mathcal{L}_1$ for all $i > 1$ and this implies $\mathbf{v}_i + \mathcal{L}_i$ lies in the translate of $\mathcal{L}_1$ under $\mathbf{v}_i$.

Since $\mathbb{Z}^d/\mathcal{L}_1$ is a group with p elements, it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let us fix an isomorphism $\mathbb{Z}^d/\mathcal{L}_1 \to \mathbb{Z}/p\mathbb{Z}$ and define $\pi \colon \mathbb{Z}^d \to \mathbb{Z}^d/\mathcal{L}_1 \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z}$ as a composition of the projection map with this isomorphism. For any $k \in \{0, \dots, p-1\}$, we define the subset of indices

$$I_k = \{i \in \{1, \dots, n\} \colon \pi(\mathbf{v}_i) = k\}.$$

The union

$$\bigcup_{i \in I_k} \{\mathbf{v}_i + \mathcal{L}_i\}$$

is equal to $\pi^{-1}(k)$, therefore it is a translate of $\mathcal{L}_1$. If for some k, $|I_k| > 1$, we have a splitting. If for all k we have $|I_k| = 1$, this means that all the lattices are isomorphic to $\mathcal{L}_1$. □

Lemma 2.4. *Assume that the lattices $\mathcal{L}_1, \ldots, \mathcal{L}_n$ have coprime determinants, i.e.*

$$\gcd(\det \mathcal{L}_1, \ldots, \det \mathcal{L}_n) = 1.$$

Then the lattice $\mathcal{L}$ generated by $\bigcup \mathcal{L}_i$, over the integers, is isomorphic to $\mathbb{Z}^d$.

Proof. Let $\mathcal{L}$ be the lattice generated by $\mathcal{L}_1, \ldots, \mathcal{L}_n$. Repeating the argument in Lemma 2.2, we see that $\det \mathcal{L} \mid \det \mathcal{L}_i$ for $1 \leq i \leq n$. Thus we have $\det \mathcal{L} \mid \gcd(\det \mathcal{L}_1, \ldots, \det \mathcal{L}_n) = 1$ and consequently $\mathcal{L} = \mathbb{Z}^d$. □

Proposition 2.5. *Let $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$ be a lattice tiling. Then for any i and j we have* $\gcd(\det \mathcal{L}_i, \det \mathcal{L}_j) > 1$.

Proof. Assume that $\gcd(\det \mathcal{L}_1, \det \mathcal{L}_2) = 1$. By Lemma 2.4, $\mathcal{L}_1 \cup \mathcal{L}_2$ generate $\mathbb{Z}^d$ and so any integer vector $\mathbf{v} \in \mathbb{Z}^d$ can be written as $\mathbf{w}_1 - \mathbf{w}_2$ for $\mathbf{w}_1 \in \mathcal{L}_1$ and $\mathbf{w}_2 \in \mathcal{L}_2$. Representing $\mathbf{v}_2 - \mathbf{v}_1$ in this form, we have $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_2 \in (\mathbf{v}_1 + \mathcal{L}_1) \cap (\mathbf{v}_2 + \mathcal{L}_2) \neq \emptyset$. We obtain a contradiction. □

2.2. **Generating functions.**

We define one of our main objects of study, namely a generating function that is attached to each lattice translate of a lattice tiling.

Definition 2.6. Let $\mathbf{v} + \mathcal{L}$ be a lattice translate, so that $\mathcal{L}$ is any integer sublattice of $\mathbb{Z}^d$ and $\mathbf{v}$ is an integer vector. We define its *generating function* by

$$\Theta_{\mathcal{L}}(\mathbf{z}) = \sum_{\substack{\mathbf{w} \in \mathcal{L} + \mathbf{v} \\ \mathbf{w} \geq 0}} \chi_{\mathbf{z}}(\mathbf{w}) := \sum_{\substack{\mathbf{w} \in \mathcal{L} + \mathbf{v} \\ \mathbf{w} \geq 0}} \mathbf{w}^{\mathbf{z}}.$$

Note the important fact that we are restricting our summation to the positive orthant. This makes the series absolutely convergent for all $\mathbf{z} = (z_1, \ldots, z_d)$ such that $|z_j| < 1$ for all $1 \leq j \leq d$. Our next step is to give an algorithm of computing Θ and to show that Θ is in fact a rational function on $\mathbb{C}^d$. To this end we introduce another definition.

Definition 2.7. Let $\mathcal{L}$ be a lattice. We define $t_1, \ldots, t_d$ as the minimal positive integers such that $(0, \ldots, 0, t_j, 0, \ldots) \in \mathcal{L}$. These integers are called the *polar values* of $\mathcal{L}$.

Example 2.8. Assume that $\mathcal{L}$ is a lattice in dimension 2 spanned by the vectors (a, b) and (c, d). Let us define $\tilde{a} = \gcd(a, c)$ and $\tilde{b} = \gcd(b, d)$, moreover $a' = a/\tilde{a}$, $c' = c/\tilde{a}$, $b' = b/\tilde{b}$ and $d' = d/\tilde{b}$. Then it is routine to see that

$$t_1 = \frac{|ad - bc|}{\tilde{a}}, \; t_2 = \frac{|ad - bc|}{\tilde{b}}.$$

It easy to see that if (a, b) and (c, d) span $\mathcal{L}$, then the determinant of $\mathcal{L}$ is $|ad - bc|$. Hence the polar values divide the determinant, a fact that we can prove in any dimension.

Lemma 2.9. *The polar values divide the determinant.*

Proof. Let $e_1 = (1, 0, \ldots, 0)$. The fact that t_1 is a polar value means that $t_1 e_1 \in \mathcal{L}$ and for any $k = 1, \ldots, t_1 - 1$, $k e_1 \notin \mathcal{L}$. Consider the subgroup of $G_{\mathcal{L}} = \mathbb{Z}^d / \mathcal{L}$ spanned by the image of e_1. We see that its order is t_1, so it divides the order of $G_{\mathcal{L}}$. □

Lemma 2.10. *Let S be the half open cube*

$$S = \{(x_1, \dots, x_d) \colon 0 \le x_i < t_i\}. \tag{2.11}$$

Then we have

$$\#(S \cap \mathcal{L}) = \frac{t_1 t_2 \dots t_d}{\det \mathcal{L}}. \tag{2.12}$$

Proof. Let $(v_1, \dots, v_d)$ be a basis of $\mathcal{L} \subset \mathbb{Z}^d$, and consider the map $J \colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$J(x_1, \dots, x_d) = x_1 v_1 + \dots + x_d v_d. \tag{2.13}$$

Then J is a linear map and $J(\mathbb{Z}^d) = \mathcal{L}$. Let $T = J^{-1}(S)$. Then T is a half-open parallelepiped with integral corners. We have

$$\#(S \cap \mathcal{L}) = \#(T \cap \mathbb{Z}^d) = \mathrm{vol}(T) = \frac{\mathrm{vol}\, S}{\det J} = \frac{t_1 \dots t_d}{\det \mathcal{L}}.$$

□

Proposition 2.14. *Let $\mathbf{v} + \mathcal{L}$ be a lattice translate and $t_1, \dots, t_d$ be the polar values of $\mathcal{L}$. Then*

$$\Theta_{\mathcal{L}}(\mathbf{z}) = \frac{R^{\mathbf{v}}(\mathbf{z})}{(1 - z_1^{t_1}) \dots (1 - z_d^{t_d})},$$

where

$$R^{\mathbf{v}}(\mathbf{z}) = \sum_{\mathbf{w} \in S \cap (\mathbf{v} + \mathcal{L})} \chi_{\mathbf{z}}(\mathbf{w}). \tag{2.15}$$

Proof. Let $\mathcal{L}_S$ be the lattice spanned by vectors $(t_1, 0, \dots, 0), \dots, (0, \dots, t_d)$. It is clear that

$$\Theta_{\mathcal{L}_S}(\mathbf{z}) = \frac{1}{(1 - z_1^{t_1}) \dots (1 - z_d^{t_d})}.$$

Now we have

$$\mathbf{v} + \mathcal{L} = \bigcup_{\mathbf{w} \in S \cap (\mathbf{v} + \mathcal{L})} \{\mathbf{w} + \mathcal{L}_S\}.$$

Hence

$$\Theta_{\mathcal{L}} = \sum_{\mathbf{w} \in S} \chi_{\mathbf{z}}(\mathbf{w}) \Theta_{\mathcal{L}_S}(\mathbf{z}) = \frac{R^{\mathbf{v}}(\mathbf{z})}{(1 - z_1^{t_1}) \dots (1 - z_d^{t_d})}.$$

□

Remark 2.16. If the lattice translate is in fact a lattice, i.e. if $\mathbf{v} = \mathbf{0}$, then we write $R(\mathbf{z})$ instead of $R^{\mathbf{v}}(\mathbf{z})$. This will be used later in Lemma 3.4.

Example 2.17. Consider a lattice $\mathcal{L} = 2\mathbb{Z} \times 2Z \cup [(1,1) + 2\mathbb{Z} \times 2\mathbb{Z}] = \{(x, y) \colon x + y = 0 \bmod 2\}$. The generating function of the lattice $2\mathbb{Z} \times 2\mathbb{Z}$ is clearly $\frac{1}{(1-x^2)(1-y^2)}$. We get therefore

$$\Theta_{\mathcal{L}}(x, y) = \frac{1 + xy}{(1 - x^2)(1 - y^2)}.$$

Example 2.18. Assume that $\mathcal{L}$ is spanned by $\mathbf{v}_1 = (4, 1)$ and $\mathbf{v}_2 = (2, 3)$. Then $t_1 = 10$ and $t_2 = 5$. In this example we see that $S \cap \mathcal{L} = \{(0,0), (4,1), (8,2), (2,3), (6,4)\}$ (the marked points on Figure 1). Hence $R(x, y) = 1 + x^4 y + x^2 y^3 + x^4 y^6 + x^4 y^6$.

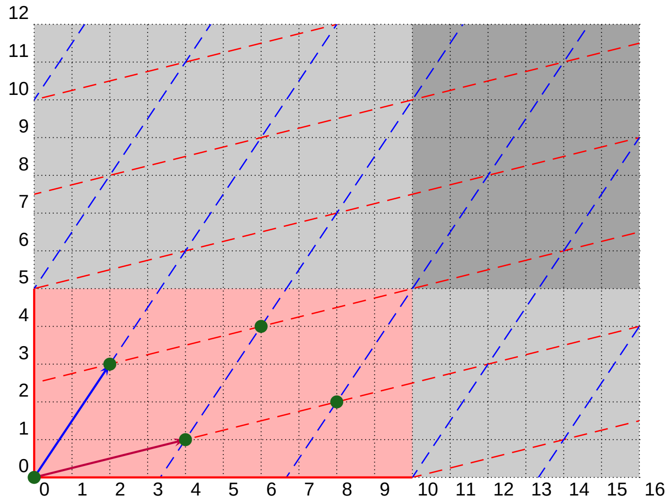

FIGURE 1. A lattice spanned by $(4,1)$ and $(2,3)$. Here the polar values are $t_1 = 10$ and $t_2 = 5$. See Example 2.18.

3. MORE ON DUAL POINTS AND CHARACTERS

Let us recall from the introduction that $\mathbf{z} \in \mathbb{C}^d$ is called a dual point of $\mathcal{L}$ if $\chi_{\mathbf{z}}(\mathbf{w}) = 1$ for all $\mathbf{w} \in \mathcal{L}$. The following "orthogonality relation" for the characters $\chi_{\mathbf{z}}$ is well-known, but as it is crucial to our proofs, we include the proof for the sake of completeness.

Lemma 3.1. *Let $\mathcal{L} \subset \mathbb{Z}^d$ be a lattice with a basis $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and the fundamental parallelepiped $\mathcal{P} = \{\lambda_1\mathbf{v}_1 + \cdots + \lambda_d\mathbf{v}_d : 0 \le \lambda_i < 1\} \subset \mathbb{R}^d$. If $\mathbf{z} = (z_1, \ldots, z_d)$ is a dual point different from $\mathbf{1} = (1, \ldots, 1)$, then*

$$\sum_{v \in \mathbb{Z}^d \cap \mathcal{P}} \chi_{\mathbf{z}}(\mathbf{v}) = 0.$$

Proof. Observe that the elements of $\mathcal{P} \cap \mathbb{Z}^d$ are in one-to-one correspondence with elements of the quotient group $\mathbb{Z}^d/\mathcal{L}$ and $\chi_{\mathbf{z}}(\mathbf{v})$ can be regarded as the evaluation of the character given by $\mathbf{z}$ on the element $\mathbf{v} \in \mathbb{Z}^d/\mathcal{L}$. This character is non-trivial because $\mathbf{z} \ne \mathbf{1}$. Now we use the standard fact that the average of a non-trivial character over a compact group (in particular over a finite group) is zero, and we are done. □

Example 3.2. Assume that $\mathbf{v}_1 = (4,1)$ and $\mathbf{v}_2 = (2,3)$. The points in P are $(0,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $(2,2)$, $(3,2)$, $(4,2)$, $(3,3)$, $(4,3)$ and $(5,3)$; see Figure 2. We consider $\mathbf{z} = (\varepsilon, \varepsilon)$, where $\varepsilon = e^{2\pi i/5}$. Then $\chi_{\mathbf{z}}$ has the following values at these points: $\varepsilon^0, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^4, \varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2$ and ε^3. The sum is clearly 0.

The next result will also be very useful for us. We first prove it for lattices, namely when the translation vector is $(0, \ldots, 0)$.

Proposition 3.3. *Suppose that $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ satisfies $z_1^{t_1} = \cdots = z_d^{t_d} = 1$ and $\mathbf{z}$ is* not *a dual point of $\mathcal{L}$ (the t_i are the polar values). Then $R(\mathbf{z}) = 0$.*

Proof. By definition $\mathbf{z}$ is a dual point of the lattice $\mathcal{L}_S$, defined to be the integral span of the vectors $(t_1, 0, \ldots, 0), \ldots,\ (0, \ldots, 0, t_d)$. The map J defined in (2.13) induces the dual map $J^* \colon \mathbb{T}^d \to \mathbb{T}^d$ by the formula

$$\chi_{J^*\mathbf{z}}(\mathbf{v}) = \chi_{\mathbf{z}}(J\mathbf{v}).$$

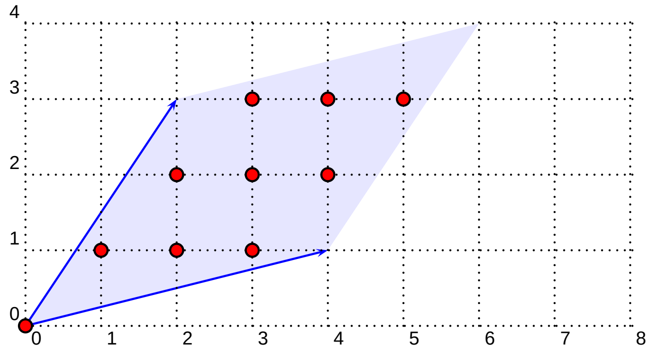

FIGURE 2. The points in the paralelliped P for the lattice spanned by $(4,1)$ and $(2,3)$.

Let $\mathbf{y} = J^*\mathbf{z}$, $\mathcal{T} = J^{-1}(\mathcal{L}_S)$ and $\mathcal{P}(\mathcal{T})$ the fundamental parallelepiped of $\mathcal{T}$. By definition, J^* maps dual points of $\mathcal{L}_S$ to dual points of $J^{-1}(\mathcal{T})$. We have

$$R(z_1,\dots,z_d) = \sum_{\mathbf{m}\in S\cap\mathcal{L}} \chi_{\mathbf{z}}(\mathbf{m}) = \sum_{\mathbf{v}\in\mathcal{P}(\mathcal{T})} \chi_{\mathbf{z}}(J\mathbf{v}) = \sum_{\mathbf{v}\in P(\mathcal{T})} \chi_{\mathbf{y}}(\mathbf{v}).$$

Now $\mathbf{y}$ is a dual point of $\mathcal{T}$, but $y \neq \mathbf{1}$, otherwise $\mathbf{z}$ would be a dual point of $\mathcal{L}$. The result follows by applying Lemma 3.1. □

We shall generalize Proposition 3.3 in the following way.

Lemma 3.4. *Let $\mathcal{L}$ be a lattice with polar values $(t_1,\dots,t_d)$. Suppose that $\mathbf{z} = (z_1,\dots,z_d)$ satisfies $z_1^{t_1} = \cdots = z_d^{t_d} = 1$. Then for any translate vector $\mathbf{v}$ we have*

$$R^{\mathbf{v}}(\mathbf{z}) = \chi_{\mathbf{z}}(\mathbf{v})R(\mathbf{z}).$$

Proof. Let S be a cube as in (2.11). We split the sum (2.15) as follows.

$$R^{\mathbf{v}}(\mathbf{z}) = \sum_{\substack{\mathbf{w}\in(\mathbf{v}+\mathcal{L})\cap S\\ \mathbf{w}-\mathbf{v}\in S}} \chi_{\mathbf{z}}(\mathbf{w}) + \sum_{\substack{\mathbf{w}\in(\mathbf{v}+\mathcal{L})\cap S\\ \mathbf{w}-\mathbf{v}\notin S}} \chi_{\mathbf{z}}(\mathbf{w}).$$

The first term is the same as

$$\chi_{\mathbf{z}}(\mathbf{v}) \sum_{\substack{\mathbf{u}\in\mathcal{L}\cap S\\ \mathbf{u}+\mathbf{v}\in S}} \chi_{\mathbf{z}}(\mathbf{u}).$$

For the second term, observe that if $\mathbf{w}-\mathbf{v}\notin S$, then there exists a unique element $\mathbf{u}'\in\mathcal{L}_S$ such that $\mathbf{w}-\mathbf{v}-\mathbf{u}'\in S$. Writing Hence we obtain

$$\sum_{\substack{\mathbf{w}\in(\mathbf{v}+\mathcal{L})\cap S\\ \mathbf{w}-\mathbf{v}\notin S}} \chi_{\mathbf{z}}(\mathbf{w}) = \sum_{\substack{\mathbf{u}\in\mathcal{L}\cap S\\ \mathbf{u}+\mathbf{v}\notin S}} \chi_{\mathbf{z}}(\mathbf{u}+\mathbf{v}-\mathbf{u}').$$

Now we note that $\chi_{\mathbf{z}}(\mathbf{u}+\mathbf{v}-\mathbf{u}') = \chi_{\mathbf{z}}(\mathbf{v})\chi_{\mathbf{z}}(\mathbf{u})\chi_{\mathbf{z}}(-\mathbf{u}')$, but $\chi_{\mathbf{z}}(-\mathbf{u}') = 1$ because of the assumptions on $\mathbf{z}$. Thus

$$R^{\mathbf{v}}(\mathbf{z}) = \chi_z(\mathbf{v}) \sum_{\substack{\mathbf{u}\in\mathcal{L}\cap S\\ \mathbf{u}+\mathbf{v}\in S}} \chi_z(\mathbf{u}) + \chi_z(\mathbf{v}) \sum_{\substack{\mathbf{u}\in\mathcal{L}\cap S\\ \mathbf{u}+\mathbf{v}\notin S}} \chi_{\mathbf{z}}(\mathbf{u})$$

and the proof is finished. □

We finish this section with another important result, whose proof follows immediately from Pontryagin duality.

Lemma 3.5. *Assume that $\mathcal{L} \subset \mathbb{Z}^d$ is a lattice and $\mathbf{v} \in \mathbb{Z}^d \setminus \mathcal{L}$. Then there exists a dual point $\mathbf{z} \in \widehat{G}_{\mathcal{L}}$ such that $\chi_{\mathbf{z}}(\mathbf{v}) \neq 1$.*

Corollary 3.6. *If we are given two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ and each dual point of $\mathcal{L}_1$ is a dual point of $\mathcal{L}_2$ then $\mathcal{L}_2 \subset \mathcal{L}_1$. In particular, if $\widehat{G}_{\mathcal{L}_1} = \widehat{G}_{\mathcal{L}_2}$ as subgroups of $\mathbb{T}^d$, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. Assume that $\mathbf{v} \in \mathcal{L}_2 \setminus \mathcal{L}_1$. Then there exist a dual point $\mathbf{z}$ of $\mathcal{L}_1$ such that $\chi_{\mathbf{z}}(\mathbf{v}) \neq 1$. But as $\mathbf{z}$ is a dual point of $\mathcal{L}_2$ by assumption, we get $\chi_{\mathbf{z}}(\mathbf{v}) = 1$, a contradiction. □

4. An explicit computation of the dual group

We recall that a lattice $\mathcal{L}$ is called *cyclic* if $\widehat{G}_{\mathcal{L}}$ is a cyclic group. In this section we first show that for any given integer lattice $\mathcal{L}$, the dual group $\widehat{G}_{\mathcal{L}}$ can be given as the zero set of a natural collection of polynomial equations, which are in fact the collection of factors in the denominator of $\Theta_{\mathcal{L}}(\mathbf{z})$. Then, in Theorem 4.5 we give a general characterization of a cyclic lattice in terms of the coordinates of its basis vectors. In the case of 2-dimensional lattice tilings, this characterization is particularly simple and will be especially useful for us in the ensuing sections of the paper.

Suppose we have a lattice $\mathcal{L} \subset \mathbb{Z}^d$, with a basis consisting of the integer vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$. Upon choosing a basis of $\mathbb{Z}^d$ we may write each vector $\mathbf{v}_k$ for $1 \le k \le d$ in the form

$$\mathbf{v}_k = (n_{1,k}, n_{2,k}, \dots, n_{d,k}).$$

We want first to characterize the characters of $\mathcal{L}$. Recall that these are elements $\mathbf{z} \in \mathbb{C}^d$, such that for any $\mathbf{v} \in \mathcal{L}$ we have $\chi_{\mathbf{z}}(\mathbf{v}) = 1$. Since $\chi_{\mathbf{z}}$ can be regarded as a homomorphism from $\mathcal{L}$ to $\mathbb{C}^*$, clearly $\mathbf{z} \in \mathbb{C}^d$ is a character if and only if for any $k = 1, \dots, d$ we have $\chi_{\mathbf{z}}(v_k) = 1$. Writing $\mathbf{z} = (z_1, \dots, z_d)$ the latter condition translates into a system of d polynomial equations in the variables $z_1, \dots, z_d$:

$$z_1^{n_{1,k}} z_2^{n_{2,k}} \dots z_d^{n_{d,k}} = 1, \tag{4.1}$$

over all $1 \le k \le d$.

We define the matrix $N = \{n_{i,j}\}$, whose columns are the basis vectors of the lattice $\mathcal{L}$, and note that N is invertible. We denote its determinant by $\det N = \Delta$. Clearly $G_{\mathcal{L}} = \mathbb{Z}^d / N \cdot \mathbb{Z}^d$. Note that changing a basis of $\mathcal{L}$ corresponds to multiplying N from the left by an unimodular matrix U_1. On the other hand, changing a basis of $\mathbb{Z}^d$ amounts to multiplying N from the right by an unimodular matrix U_2. Clearly the groups $\mathbb{Z}^d / N \cdot \mathbb{Z}^d$ and $\mathbb{Z}^d / U_1 N U_2 \cdot \mathbb{Z}^d$ are isomorphic.

Let $M = \{M_{i,j}\}$ be the adjugate matrix of N, namely the matrix defined by the relation $M^T = \Delta \cdot N^{-1}$. The entries of M are the determinants of the various $(d-1) \times (d-1)$ minors of N, and are in this case integers.

Lemma 4.2. *There are exactly Δ distinct solutions to the system of polynomial equations (4.1), and they are given by*

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{\Delta}\left(l_1 M_{1,1} + \cdots + l_d M_{1,d}\right)} \\ e^{\frac{2\pi i}{\Delta}\left(l_1 M_{2,1} + \cdots + l_d M_{2,d}\right)} \\ \vdots \\ e^{\frac{2\pi i}{\Delta}\left(l_1 M_{d,1} + \cdots + l_d M_{d,d}\right)} \end{pmatrix}$$

as the l_d vary over the integers.

Proof. We take logarithms of both sides of (4.1), where we use the multi-valued complex log function, which gives us $\log(1) = 0 + 2\pi i l$, for all $l \in \mathbb{Z}$. We thus get, for each $1 \le k \le d$:

$$n_{1,k} \log(z_1) + n_{2,k} \log(z_2) \dots n_{d,k} \log(z_d) = 2\pi i l_d \tag{4.3}$$

We must take into account all possible branches of the multi-valued log function when applying the log to a product of complex numbers, so that using $\log(z_d) = \mathrm{Log}(z_d) + 2\pi i m_d$ gives us:

$$\begin{pmatrix} n_{1,1}\mathrm{Log}(z_1) + n_{1,1}(2\pi i m_1) & + \cdots + & n_{d,1}\mathrm{Log}(z_d) + n_{d,1}(2\pi i m_d) \\ n_{1,2}\mathrm{Log}(z_1) + n_{1,2}(2\pi i m_1) & + \cdots + & n_{d,2}\mathrm{Log}(z_d) + n_{d,2}(2\pi i m_d) \\ \vdots & \vdots & \vdots \\ n_{1,d}\mathrm{Log}(z_1) + n_{1,d}(2\pi i m_1) & + \cdots + & n_{d,d}\mathrm{Log}(z_d) + n_{d,d}(2\pi i m_d) \end{pmatrix} = \begin{pmatrix} 2\pi i l_1 \\ \vdots \\ 2\pi i l_d \end{pmatrix}. \tag{4.4}$$

We may rewrite (4.4) as follows:

$$\begin{pmatrix} \mathrm{Log}(z_1) + 2\pi i m_1 \\ \mathrm{Log}(z_2) + 2\pi i m_2 \\ \vdots \\ \mathrm{Log}(z_d) + 2\pi i m_d \end{pmatrix} = \frac{2\pi i}{\Delta} M \begin{pmatrix} l_1 \\ \vdots \\ l_d \end{pmatrix}.$$

We finally arrive at $\mathrm{Log}(z_k) + 2\pi i m_k = \frac{2\pi i}{\Delta}\left(l_1 M_{k,1} + \cdots + l_d M_{k,d}\right)$, so that

$$z_k = e^{\frac{2\pi i}{\Delta}\left(l_1 M_{k,1} + \cdots + l_d M_{k,d}\right)}.$$

□

Theorem 4.5. *$\mathcal{L}$ is a cyclic lattice if and only if* $\gcd(M_{1,1}, M_{1,2}, \dots, M_{d,d}) = 1$

Proof. Using Smith normal form for the ring of integer matrices, we can find two unimodular matrices U_1 and U_2 such that $U_1 N U_2 = A$ is a diagonal matrix with diagonal entries satisfying $a_{i+1} \mid a_i$ for $1 \le i \le d-1$. It is moreover known that the product $a_{d-k+1} \cdots a_d$ is equal to the greatest common divisor of all $k \times k$ minors of the matrix N for all $1 \le k \le d$. The lattice generated by A now has dual group $\mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \cdots \oplus \mathbb{Z}_{a_d}$, which is isomorphic to $\widehat{G}_{\mathcal{L}}$. Thus, $\mathcal{L}$ is cyclic if and only if $a_2 = 1$ and this is equivalent to the arithmetic condition $\gcd(M_{1,1}, \dots, M_{d,d}) = 1$. □

In particular, it follows immediately from Theorem 4.5 that if $\mathcal{L} \subset \mathbb{Z}^2$ is spanned by (a, b) and (c, d), then it is cyclic if and only if $\gcd(a, b, c, d) = 1$.

5. Residue calculus

5.1. **Residues of generating functions.** Assume now that we have lattices $\mathcal{L}_1, \dots, \mathcal{L}_n$ in $\mathbb{Z}^d$. Denote the polar values of the lattice $\mathcal{L}_i$ by $t_{i,1}, \dots, t_{i,d}$. Assume that there exist non-negative vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that we have the lattice tiling

$$\begin{cases} \{\mathbf{v}_1 + \mathcal{L}_1\} \cup \{\mathbf{v}_2 + \mathcal{L}_2\} \cup \cdots \cup \{\mathbf{v}_n + \mathcal{L}_n\} = \mathbb{Z}^d, & \\ \{\mathbf{v}_i + \mathcal{L}_i\} \cap \{\mathbf{v}_j + \mathcal{L}_j\} = \emptyset & \text{for } i \ne j. \end{cases} \tag{5.1}$$

Let Θ_i be a generating function of $\mathbf{v}_i + \mathcal{L}_i$.

Lemma 5.2. *We have the following equality of rational functions, with $\mathbf{z} = (z_1, \dots, z_d)$:*

$$\sum_{i=1}^{n} \Theta_i(\mathbf{z}) = \frac{1}{(1 - z_1) \cdots (1 - z_d)}. \tag{5.3}$$

Moreover, the identity (5.3) *is equivalent to the condition that* $\mathbf{v}_1+\mathcal{L}_1, \ldots, \mathbf{v}_n+\mathcal{L}_n$ *provide a lattice tiling of* $\mathbb{Z}^d$*.*

Proof. The right hand side of (5.3) is the generating function for the tiling consisting of a single lattice $\mathbb{Z}^d$. Let us choose $\mathbf{z}$ such that $|z_j|<1$ for all j. By the tiling condition we have

$$\sum_{\substack{\mathbf{w}\in\mathbb{Z}^d\\ \mathbf{w}\geq 0}} \chi_{\mathbf{z}}(\mathbf{w}) = \sum_{j=1}^{n} \sum_{\substack{\mathbf{w}\in\mathbf{v}_j+\mathcal{L}_j\\ \mathbf{w}\geq 0}} \chi_{\mathbf{z}}(\mathbf{w}), \tag{5.4}$$

where we use the fact that we can change the summation order of absolutely convergent power series. Equation (5.4) is equivalent to (5.3) for $\mathbf{z}$ such that $|z_j|<1$, see Definition 2.6. Both sides of (5.3) are rational functions agreeing on an open subset of $\mathbb{C}^d$, so they are equal. It is also clear that (5.4) implies the tiling condition. □

We now fix some $\mathbf{z}\in\mathbb{C}^d$. Consider a torus $T=\{\mathbf{u}\in\mathbb{C}^d\colon |z_1-u_1|=\ldots=|z_d-u_d|=\varepsilon\}$ for $\varepsilon>0$ sufficiently small. Equation (5.3) implies that

$$\sum_{j=1}^{n} \int_T \Theta_j(u)d\mathbf{u} = \int_T \frac{d\mathbf{u}}{(1-u_1)\ldots(1-u_d)}. \tag{5.5}$$

Here $d\mathbf{u}=du_1\wedge\ldots\wedge du_d$ is the volume form on T (note that (5.5) can be regarded as comparison of multidimensional residues, see [12]). We want to study the integrals appearing on the right hand side of (5.5). To this end recall that by Proposition 2.14 that we can write

$$\Theta_i(\mathbf{z}) = \frac{R_i(\mathbf{z})}{(1-z_1^{t_{i,1}})\cdots(1-z_d^{t_{i,d}})},$$

where

$$R_i(\mathbf{z}) = \sum_{\mathbf{v}\in S_i\cap\mathbf{v}_i+\mathcal{L}_i} \chi_{\mathbf{z}}(\mathbf{v}).$$

Lemma 5.6. *For any* $j=1,\ldots,n$*, the integral*

$$\int_T \frac{R_j(\mathbf{u})}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,k}})}d\mathbf{u}$$

is zero unless $\mathbf{z}\in\widehat{G}_{\mathcal{L}_j}$*. In the latter case it is equal to* $\frac{(-2\pi i)^d}{\det L_j}\chi_{\mathbf{z}}(\mathbf{v}_j+\mathbf{1})$*.*

Proof. As R_j is analytic at $\mathbf{z}$, we have

$$\int_T \frac{R_j(\mathbf{u})}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})}d\mathbf{u} = R_j(\mathbf{z})\int_T \frac{d\mathbf{u}}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})}. \tag{5.7}$$

But

$$\int_T \frac{d\mathbf{u}}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})} = \prod_{k=1}^{d}\int_{|u_k-z_k|=\varepsilon} \frac{du_k}{1-u_k^{t_{j,k}}}.$$

By Goursat's lemma the integrals on the right hand side vanish unless $z_1^{t_{j,1}}=\cdots=z_d^{t_{j,d}}=1$. So assume that $z_1^{t_{j,1}}=\cdots=z_d^{t_{j,d}}=1$. Since for any $x_0\in\mathbb{C}$ and any integer $m>0$

$$\int_{|x-x_0|=\varepsilon} \frac{dx}{1-x^m} \overset{x=zx_0}{=} z\int_{|z-1|=\varepsilon} \frac{dz}{1-z^m} = -2\pi i\frac{x_0}{m},$$

we have

$$\prod_{k=1}^{n}\int_{|u_k-z_k|=\varepsilon}\frac{du_k}{1-u_k^{t_{j,k}}}=\frac{(-2\pi i)^d}{t_{j,1}\dots t_{j,d}}z_1\dots z_d. \tag{5.8}$$

Given that $z_1^{t_{j,1}}=\dots=z_d^{t_{j,d}}=1$, to compute $R_j(\mathbf{z})$ in this case we use Proposition 3.3 and Lemma 3.4. We get $R_j(\mathbf{z})=0$ unless $\mathbf{z}$ is a dual point of L_j. In this case, by Lemma 3.4 we have

$$R_j(\mathbf{z})=\frac{t_{j,1}\dots t_{j,d}}{\det\mathcal{L}_j}\chi_{\mathbf{z}}(\mathbf{v}_j). \tag{5.9}$$

Substituting the R_j from (5.9) and the integral from (5.8) into (5.7) we conclude the proof. □

We now combine Lemma 5.6 with Lemma 5.2 to obtain our main technical result.

Proposition 5.10. *Let $\mathbf{z}$ be a dual point of $\mathcal{L}_1$. Then we have*

$$\sum_{j\colon \mathbf{z}\in\widehat{G}_{\mathcal{L}_j}}\frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det\mathcal{L}_j}=\begin{cases}1, & \text{if } \mathbf{z}=\mathbf{1}\\ 0 & \text{otherwise.}\end{cases} \tag{5.11}$$

Proof. Consider (5.5). On the right hand side we obtain 0 if $\mathbf{z}\neq\mathbf{1}$; if $\mathbf{z}=\mathbf{1}$, the integral is $(-2\pi i)^d$. The integral on the left hand side is computed using Lemma 5.6. The proposition follows immediately. □

From Proposition 5.10 we can deduce an immediate corollary.

Corollary 5.12. *Assume that $\mathbf{z}$ is a dual point of $\mathcal{L}_i$ and $\mathbf{z}\neq(1,\dots,1)$. Then there exists at least one other $\mathcal{L}_j$ such that $\mathbf{z}$ is also a dual point of $\mathcal{L}_j$.*

Proof. If $\mathbf{z}$ belongs only to $\widehat{G}_{\mathcal{L}_i}$, then the right hand side of (5.11) is $\frac{\chi_{\mathbf{z}}(\mathbf{v}_i)}{\det\mathcal{L}_i}\neq 0$, and we obtain a contradiction. □

Example 5.13. Consider the four lattices $\mathcal{L}_1=(2\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{Z})$, $\mathcal{L}_2=(2\mathbb{Z}\times\mathbb{Z}\times 2\mathbb{Z})$, $\mathcal{L}_3=(\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z})$ and $\mathcal{L}_4=(2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z})\cup(1,1,1)+((2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z})$. It is known that $(1,0,0)+\mathcal{L}_1$, $(0,0,1)+\mathcal{L}_2$, $(0,1,0)+\mathcal{L}_3$ and $\mathcal{L}_4$ tile $\mathbb{Z}^3$. We have

$$\frac{z_1}{(1-z_1^2)(1-z_2^2)(1-z_3)}+\frac{z_3}{(1-z_1^2)(1-z_2)(1-z_3^2)}+$$
$$+\frac{z_2}{(1-z_1)(1-z_2^2)(1-z_3^2)}+\frac{1+z_1z_2z_3}{(1-z_1^2)(1-z_2^2)(1-z_3^2)}=$$
$$=\frac{1}{(1-z_1)(1-z_2)(1-z_3)}.$$

The dual point are respectively

$$\begin{aligned}
L_1&\colon\ (1,-1,1),\ (-1,1,1),\ (-1,-1,1)\\
L_2&\colon\ (1,1,-1),\ (-1,1,1),\ (-1,1,-1)\\
L_3&\colon\ (1,1,-1),\ (1,-1,1),\ (1,-1,-1)\\
L_4&\colon\ (1,-1,-1),\ (-1,1,-1),\ (-1,-1,1).
\end{aligned}$$

We see that each point $(\pm1,\pm1,\pm1)$ different from $(1,1,1)$ and $(-1,-1,-1)$ occurs precisely twice.

Corollary 5.14. *Assume that the point $\mathbf{z}\neq\mathbf{1}$ is a dual point of the lattice $\mathcal{L}_i$ and $\mathcal{L}_j$ and no other lattice. Then* $\det\mathcal{L}_i=\det\mathcal{L}_j$.

Proof. By (5.11) we get

$$\frac{\chi_{\mathbf{z}}(\mathbf{v}_i)}{\det \mathcal{L}_i} + \frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det \mathcal{L}_j} = 0.$$

But the numerators are roots of unity, so the denominators, both being positive integers, must agree. □

Corollary 5.15. *We have the following relation*

$$\sum_{i=1}^{n} \frac{1}{\det \mathcal{L}_i} = 1.$$

Proof. Apply (5.11) with $\mathbf{z} = \mathbf{1}$. □

This "density" result, which shows that the sum of the densities is always 1, can also be proved in an elementary way. Namely, we observe that the number of integer points of $\mathcal{L}_i$ in a cube $[-N, N]^d$ is equal to $(2N)^d / \det \mathcal{L}_i + O(N^{d-1})$. We can now prove Theorem 1.6 from the introduction.

Proof of Theorem 1.6. Indeed, let $\mathbf{z}$ be an element in $\widehat{G}_{\mathcal{L}_1}$ of maximal order equal to $\det \mathcal{L}_1$. By Proposition 5.12 there exists $j > 1$ such that $\mathbf{z} \in \widehat{G}_{\mathcal{L}_j}$. But then the whole group spanned by $\mathcal{L}_1$ lies in $\widehat{G}_{\mathcal{L}_j}$, hence $\widehat{G}_{\mathcal{L}_1} \subset \widehat{G}_{\mathcal{L}_j}$. By maximality of $\det \mathcal{L}_1$ we have $\widehat{G}_{\mathcal{L}_1} = \widehat{G}_{\mathcal{L}_j}$. Now Corollary 3.6 implies that $\mathcal{L}_1 = \mathcal{L}_j$. □

5.2. **Sufficiency of the residue condition.** We will show now that the conditions given by Proposition 5.10 are sufficient for the tiling. More precisely, we have the following result.

Theorem 5.16. *Let $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$ be translate lattices in $\mathbb{Z}^d$ with generating functions $\Theta_1, \ldots, \Theta_n$. Suppose that for any $\mathbf{z} \in \mathbb{T}^d$ we have the relation (5.11) then the lattices tile $\mathbb{Z}^d$.*

Remark 5.17. If $\mathbf{z}$ is not a dual point of any of lattices, then (5.11) is an empty relation. Therefore it is enough to check (5.11) only in finitely many places.

Proof. We shall strive to prove that $\Theta_1, \ldots, \Theta_n$ satisfy (5.3). In the following, for a polynomial P in variables $z_1, \ldots, z_d$, we write $\deg_k P$ as the degree in variable k.

Observe that each generating function Θ_j vanishes at infinity. More precisely, if we fix $z_1, \ldots, \widehat{z}_k, \ldots, z_d$ such that for any $m \neq k$, $z_m^{t_{mj}} \neq 1$ (where t_{mj} denotes the m-th polar value of Θ_j), we have $\lim_{z_k \to \infty} \Theta_j(z_1, \ldots, z_d) = 0$. This is a direct consequence of the fact, that $\deg_k R_j^{\mathbf{v}} < t_{mk}$. In particular if we define

$$\Theta(\mathbf{z}) = \Theta_1(\mathbf{z}) + \cdots + \Theta_n(\mathbf{z}) - \frac{1}{(1 - z_1) \ldots (1 - z_d)},$$

then Θ also vanishes at infinity. We can write Θ in the following way.

$$\Theta(\mathbf{z}) = \frac{R(\mathbf{z})}{Q_1(z_1) \ldots Q_d(z_d)},$$

where R is a polynomial in $z_1, \ldots, z_d$ and Q_m is the least common multiplier of $z_m^{t_{m1}} - 1, \ldots, z_m^{t_{md}} - 1$. Notice that Q_m is square free. The asymptotics of Θ implies that

$$\deg_k R < \deg Q_k. \tag{5.18}$$

Now we use the residue condition (5.3). We claim that if $u_1, \ldots, u_d$ are such that $Q_1(u_1) = \cdots = Q_d(u_d) = 0$, then $R(u_1, \ldots, u_d) = 0$. In fact, $R(u_1, \ldots, u_d)$ is then proportional to the residue of the form $\Theta(\mathbf{z}) d\mathbf{z}$ at $u_1, \ldots, u_d$. But this residue is zero by the assumption of the theorem. In fact, if $\mathbf{u} = (u_1, \dots, u_d)$ is not a dual point of $\mathcal{L}_j$, then the residue of $\Theta_j(\mathbf{z})d\mathbf{z}$ at $\mathbf{u}$ is zero (see proof of Lemma 5.6), if it is a dual point of some lattices, then we use (5.11).

By induction we shall show that for any k, and any $u_{k+1}, \dots, u_d$ such that $Q_{k+1}(u_{k+1}) = \dots = Q_d(u_d) = 0$ we have

$$R(z_1, \dots, z_k, u_{k+1}, \dots, u_d) \equiv 0 \text{ as a polynomial in } z_1, \dots, z_k$$

The induction assumption (for $k = 0$) is done. Now suppose we have proved it for $k-1$. The polynomial

$$P_k(z_1, \dots, z_{k-1}, z_k) := R(z_1, \dots, z_k, u_{k+1}, \dots, u_d)$$

vanishes for $z_k = u_k$ for any u_k such that $Q_k(u_k) = 0$. Hence P_k is divisible by $(z_k - w_1) \cdot \ldots \cdot (z_k - w_l)$, where $w_1, \dots, w_l$ are roots of Q_k. Since Q_k is square free, we have $l = \deg Q_k$. But $\deg_k P_k < \deg Q_k$. The only possibility is that $P_k \equiv 0$.

The statement for $k = d$ implies that R is identically zero. This is equivalent to (5.3), and the proof is finished. □

Remark 5.19. The above proof is a generalization of the fact that if a rational function on $\mathbb{C}$ has only simple poles, vanishes at infinity and has residue 0 at each pole, then it is equal to zero everywhere. One could express the above proof in the language of multidimensional residues, but we wanted the proofs to be accessible to non-experts.

6. Dual points and lattice tilings in dimension 2

6.1. **Characterisation of dual groups in dimension 2.** Theorem 4.5 tells us that a two-dimensional lattice $\mathcal{L}$ generated by $v_1 = (a, b)$ and $v_2 = (c, d)$ is cyclic if and only if $\gcd(a, b, c, d) = 1$. The quantity $e = \gcd(a, b, c, d)$ does not depend on the choice of basis and we will call it the *multiplicity* of $\mathcal{L}$.

It follows from the Smith Normal Form (see Section 4) that we have an isomorphism, namely $\widehat{G}_{\mathcal{L}} = \mathbb{Z}_e \oplus \mathbb{Z}_{\frac{\det \mathcal{L}}{e}}$. Furthermore $e^2 | \det \mathcal{L}$. In particular, if $\det \mathcal{L}$ is square free, the lattice is necessarily cyclic.

In more than two dimensions, the cyclicity is more subtle and the groups $\widehat{G}_{\mathcal{L}}$ might be more complicated. For example, the lattice spanned by $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$ has the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

To stress the difference between the two-dimensional case and the higher dimensional one, we first prove a simple result.

Lemma 6.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two sublattices of $\mathbb{Z}^2$ with equal multiplicities $e_1 = e_2 = e$. Assume that there is a common vector $w = (a, b) \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that $\gcd(a, b) = e$. Assume also that $\det \mathcal{L}_1 = \det \mathcal{L}_2$, then we have $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. Rescaling the two lattices by the factor $1/e$ we can assume that $e = 1$. Let $\mathbf{w} = (a, b)$ and $\mathbf{v}_1 = (c_1, d_1)$, $\mathbf{v}_2 = (c_2, d_2)$ be two vectors such that $(\mathbf{w}, \mathbf{v}_i)$ spans $\mathcal{L}_i$ and $ad_i - bc_i = \det \mathcal{L}_i$, $i = 1, 2$. The equality of determinants implies that

$$a(d_1 - d_2) = b(c_1 - c_2).$$

As $\gcd(a, b) = 1$, we infer that there exists $k \in \mathbb{Z}$ such that $c_1 - c_2 = ka$, $d_1 - d_2 = kb$. This means that $\mathbf{v}_2 = \mathbf{v}_1 + k\mathbf{w}$. In particular $\mathbf{v}_2 \in \mathcal{L}_1$, so $\mathcal{L}_2 \subset \mathcal{L}_1$. Precisely the same arguments prove that $\mathcal{L}_1 \subset \mathcal{L}_2$. □

Remark 6.2. The proof does not work if we do not assume that $\gcd(a, b) = e$. For example consider $\mathcal{L}_1 = 2\mathbb{Z} \times \mathbb{Z}$ and $\mathcal{L}_2 = \mathbb{Z} \times 2\mathbb{Z}$. Then $(2, 2) \in \mathcal{L}_1 \cap \mathcal{L}_2$ and $\det \mathcal{L}_1 = \det \mathcal{L}_2 = 2$, but $\mathcal{L}_1$ and $\mathcal{L}_2$ are different. In the language of dual points and dual groups we can reformulate Lemma 6.1 as follows.

Corollary 6.3. *Let us be given two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ with the same multiplicity e and determinant Δ. Assume that there exists $g_1 \in \widehat{G}_{\mathcal{L}_1}$ and $g_2 \cap \widehat{G}_{\mathcal{L}_2}$, both of order Δ/e, such that $g_1^e = g_2^e$. Then $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. Let $g = g_1^e = g_2^e = (z_1, z_2) \in \mathbb{T}^2$. Consider the two integer lattices $\frac{1}{e}\mathcal{L}_1$ and $\frac{1}{e}\mathcal{L}_1$. They both are cyclic, have determinant Δ/e^2 and admit g as a common dual point. Now g also has order Δ/e^2, hence it is a generator for both $\widehat{G}_{\frac{1}{e}\mathcal{L}_1}$ and $\widehat{G}_{\frac{1}{e}\mathcal{L}_2}$. So we must have $\widehat{G}_{\frac{1}{e}\mathcal{L}_1} = \widehat{G}_{\frac{1}{e}\mathcal{L}_2}$. The result follows easily by applying Corollary 3.6 and then rescaling to the original lattices. □

6.2. **Tilings in dimension 2.** In this subsection we assume that we are given lattices $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$, which provide a lattice tiling of $\mathbb{Z}^2$. Let us reorder the lattices in the following way.

(a) If $i < j$, then $\frac{1}{e_i}\det\mathcal{L}_i \geq \frac{1}{e_j}\det\mathcal{L}_j$. In other words, the maximal cyclic subgroup of $\widehat{G}_{\mathcal{L}_i}$ has at least the same order as the maximal cyclic subgroup of $\widehat{G}_{\mathcal{L}_j}$.
(b) If $i < j$, but $\frac{1}{e_i}\det\mathcal{L}_i = \frac{1}{e_j}\det\mathcal{L}_j$, then $\det\mathcal{L}_i \geq \det\mathcal{L}_j$.

Theorem 6.4. *If the number e_1 is of the form p^r for p a prime, then the tiling has the translation property.*

Proof. We set $\alpha = \det\mathcal{L}_1/e_1$. Let $\mathbf{z} \in \widehat{G}_{\mathcal{L}_1}$ be an element of order α. We define a sequence $1 = n_1 < n_2 < \cdots < n_s$ of indices such that $\mathbf{z}$ belongs to $\widehat{G}_{\mathcal{L}_{n_1}}, \ldots, \widehat{G}_{\mathcal{L}_{n_s}}$ and to no other lattice. To shorten the notation we will write $\mathcal{L}_k^{\mathbf{z}}$, $\mathbf{v}_k^{\mathbf{z}}$ instead of $\mathcal{L}_{n_1}$, $\mathbf{v}_{n_1}$.

The maximum order over all elements in $\widehat{G}_{\mathcal{L}_k^{\mathbf{z}}}$ is $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}}$. By the ordering condition we have $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}} \leq \alpha$. But $\widehat{G}_{\mathcal{L}_k^{\mathbf{z}}}$ contains the element $\mathbf{z}$ of order α. Hence $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}} = \alpha$.

Lemma 6.5. *There cannot be two indices n_k and n_l, $n_k \neq n_l$ such that $e_k^{\mathbf{z}} = e_l^{\mathbf{z}}$.*

Proof. If this were the case, the lattices $\mathcal{L}_k^{\mathbf{z}}$ and $\mathcal{L}_l^{\mathbf{z}}$ would have the same determinant and multiplicity and would share an element $\mathbf{z}$ of order equal to the order of each lattice. By Corollary 6.3 we obtain that $\mathcal{L}_k^{\mathbf{z}} = \mathcal{L}_l^{\mathbf{z}}$, so there exists a translate. □

Let us apply now Proposition 5.10 to get the following equation

$$\sum_{k=1}^{s} \frac{\chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}})}{\alpha e_k^{\mathbf{z}}} = 0, \tag{6.6}$$

where we wrote $\det\mathcal{L}_k^{\mathbf{z}} = \alpha e_k^{\mathbf{z}}$. Now the expression $\chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}})$ is a root of unity. Let us note

$$a_k = \chi_{\mathbf{z}}(\mathbf{v}_1^{\mathbf{z}})^{-1}\chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}}).$$

We denote by g the minimal positive integer such that $a_k^g = 1$ for all $k = 1, \ldots, s$. Equation (6.6) takes the following form:

$$\sum_{k=1}^{s} \frac{a_k}{e_k^{\mathbf{z}}} = 0. \tag{6.7}$$

We use now the following lemma.

Lemma 6.8. *Suppose that there exists a prime q an integer $l > 0$, and an index $k \in \{1, \ldots, s\}$ such that $q^l | e_k^{\mathbf{z}}$. Then there exists $k' \in \{1, \ldots, s\}$, $k' \neq k$ such that $q^l | e_{k'}^{\mathbf{z}}$.*

Proof of Lemma 6.8. Assume the contrary, so that there exists a unique k such that $q^l | e^{\mathbf{z}}_k$. Let B be the least common multiplier of $e^{\mathbf{z}}_1, \dots, e^{\mathbf{z}}_s$ and $B_k = B/e^{\mathbf{z}}_k$. By assumptions of the lemma, for any $n \neq k$ we have $q | B_n$ and $q \not| B_k$.

Equation (6.7) can be now rewritten as

$$B_k + \sum_{n \neq k} B_m \varepsilon^{\gamma_n} = 0, \tag{6.9}$$

where ε is a root of unity of order g, $\gamma_n \in \{0, \dots, g-1\}$ and a_n from (6.7) is equal to $\varepsilon^{\gamma_n - \gamma_j} a_j$. The expression on the left hand side is a polynomial in ε. Let us denote this polynomial by $P(\varepsilon)$. By assumption on $B_1, \dots, B_s$ we have.

$$P(\varepsilon) = B_k + qQ(\varepsilon),$$

where Q is a polynomial with integer coefficients.

Now, let H be the minimal integer polynomial for a g-th root of unity. It is a monic, symmetric polynomial. Since $P(\varepsilon) = 0$, H divides P. Since H is monic, the quotient $R = P/H \in \mathbb{Z}[x]$ has integer coefficients. We end up with the following relation in the ring $\mathbb{Z}[x]$:

$$B_k + qQ(x) = R(x)H(x). \tag{6.10}$$

Let us now reduce this equation modulo q. We get $B_k \bmod q = (R \bmod q)(H \bmod q)$, where $H \bmod q$ has positive degree (because H is monic) and $B_k \not\equiv 0 \bmod q$. This cannot hold, for either $\deg(R \bmod q) \geq 0$ and the r.h.s. has positive degree, or $R \equiv 0 \bmod q$ so the l.h.s. must be zero. □

Remark 6.11. We point out that Lemma 6.8 works without any assumption on the existence of a translate. It is a direct consequence of Proposition 5.10, that is of the tiling condition. The result is valid also in higher dimensions, if we define e as the quotient of determinant over the order.

Conclusion of the proof of Theorem 6.4

We apply now Lemma 6.8 to $q^l = p^r = e^{\mathbf{z}}_1 = e_1$. We find another index $k > 1$ such that $p^r | e^{\mathbf{z}}_k$. But $e^{\mathbf{z}}_k \leq e_1 = p^r$ by the tiling condition, so we must have $e^{\mathbf{z}}_k = e_1$. But this is excluded by Lemma 6.5.

□

6.3. **Proof of Theorem 1.5.** Suppose that $\mathbf{v}_1 + \mathcal{L}_1, \dots, \mathbf{v}_n + \mathcal{L}_n$ provide a tiling of $\mathbb{Z}^d$. For each $1 \leq t \leq n$, we denote by g_t the maximal order of all of the elements in $\mathcal{L}_t$. We also set $e_t = \det \mathcal{L}_t / g_t$.

Suppose p^k with $k > 0$ divides $\det \mathcal{L}_t$. It follows that $p^a | g_t$ and $p^b | e_t$ for some non-negative integers such that $a + b = k$. It is easy to see that $a > 0$. Let us choose an element $\mathbf{z} \in \widehat{G}_{\mathcal{L}_t}$ of order p^a and assume that $\mathbf{z} \in \widehat{G}^{\mathbf{z}}_{\mathcal{L}_k}$ with $k = 1, \dots, r$ for some r. Proposition 5.10 implies that

$$\sum_{j=1}^{r} \frac{\chi_{\mathbf{z}}(\mathbf{v}^{\mathbf{z}}_j)}{\det \mathcal{L}^{\mathbf{z}}_j} = 0.$$

We act as in the proof of Lemma 6.8. We write $\chi_{\mathbf{z}}(\mathbf{v}^{\mathbf{z}}_j) = \varepsilon^{\gamma_j}$ for some root of unity of order p^a, multiply the equation by the least common multiplier of the $\det \mathcal{L}^{\mathbf{z}}_j$'s and reduce modulo p. The theorem quickly follows. □

7. Translation-free tilings with a few lattices

We call a tiling *translation-free* if it does not have the translation property. In this section we show that the counterexample in dimension 3 given in [6] is minimal.

Lemma 7.1. *There is no* translation-free *tiling with three or less lattices.*

Proof. Obviously, there is no *translation-free* tiling with two lattices. Suppose now that there is a *translation-free* tiling with three lattices $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$. We may assume that $\det \mathcal{L}_1 \geq \det \mathcal{L}_2 \geq \det \mathcal{L}_3$. By Corollary 5.15, we have

$$\frac{1}{\det \mathcal{L}_1} + \frac{1}{\det \mathcal{L}_2} + \frac{1}{\det \mathcal{L}_3} = 1.$$

If $\det \mathcal{L}_3 = 2$, then $\mathbf{v}_1 + \mathcal{L}_1$ and $\mathbf{v}_2 + \mathcal{L}_2$ provide a tiling of $2\mathbb{Z} \times \mathbb{Z}^{d-1}$ by Proposition 2.3, hence $\mathcal{L}_1 = \mathcal{L}_2$. So suppose $\det \mathcal{L}_3 = 3$. But then $\det \mathcal{L}_1 = 3$ and $\det \mathcal{L}_2 = 3$, in particular the group $\widehat{G}_{\mathcal{L}_1}$ is cyclic. By Theorem 1.6 there must exist a translate. □

Proposition 7.2. *A* translation-free *tiling with four lattices has* $\det \mathcal{L}_1 = \det \mathcal{L}_2 = \det \mathcal{L}_3 = \det \mathcal{L}_4 = 4$.

Proof. Again assume that $\det \mathcal{L}_1 \geq \det \mathcal{L}_2 \geq \det \mathcal{L}_3 \geq \det \mathcal{L}_4$. As before, if $\det \mathcal{L}_4 = 2$, then $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ tile $2\mathbb{Z} \times \mathbb{Z}^{d-1}$, and this tiling is non-special by Lemma 7.1. Assume that $\det \mathcal{L}_4 = 3$. By Proposition 2.5, 3 must divide $\det \mathcal{L}_1, \det \mathcal{L}_2$ and $\det \mathcal{L}_3$. Let $d_i = \frac{1}{3} \det \mathcal{L}_i$. We must have

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 2,$$

which is possible only if $d_3 = 1$ and $d_1 = d_2 = 2$. But then the group $\widehat{G}_{\mathcal{L}_1}$ has order 6, hence it is cyclic, so there is a translate.

Now since we have shown that for $i = 1, \dots 4$, $\det \mathcal{L}_i \geq 4$ we have

$$\frac{1}{\det \mathcal{L}_1} + \frac{1}{\det \mathcal{L}_2} + \frac{1}{\det \mathcal{L}_3} + \frac{1}{\det \mathcal{L}_4} \leq 1$$

with an equality $\det \mathcal{L}_1 = \det \mathcal{L}_2 = \cdots = \det \mathcal{L}_4 = 4$. □

We can fully classify all *translation-free* tilings with four lattices. It turns out that the lattice tiling given in Example 5.13, which appeared in [6], completely captures all translation-free tilings of $\mathbb{Z}^d$ with exactly 4 lattice translates.

Proposition 7.3. *Suppose $\mathcal{T}$ is a lattice tiling of $\mathbb{Z}^d$ with exactly 4 lattice translates, and $d \geq 3$. Then $\mathcal{T}$ does not have the translation property if and only if it is given by the lattice translates:*

$$\mathcal{L}_1 = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{d-3} + (1, 0, 0, 0, \dots, 0),$$
$$\mathcal{L}_2 = 2\mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}^{d-3} + (0, 0, 1, 0, 0, \dots, 0),$$
$$\mathcal{L}_3 = \mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}^{d-3} + (0, 1, 0, 0, 0, \dots, 0),$$

and

$$\mathcal{L}_4 = 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}^{d-3} \cup \{2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}^{d-3} + (1, 1, 1, 0, 0, \dots, 0)\}.$$

Proof. We know that $\det \mathcal{L}_1 = \cdots = \det \mathcal{L}_4 = 4$. If any of the groups $\widehat{G}_{\mathcal{L}_1}, \dots, \widehat{G}_{\mathcal{L}_4}$ is cyclic, then by Theorem 1.6 we conclude that the tiling is translation-free. Hence, all the groups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let g_1 and g_2 generate $\widehat{G}_{\mathcal{L}_1}$. By Proposition 5.12 we infer that there exist $j > 1$ such that $g_1 \in \widehat{G}_{\mathcal{L}_j}$. We shall first show that this j is unique. Indeed, the three elements g_1, g_2 and $g_1 + g_2$ have the property that any two generate $\widehat{G}_{\mathcal{L}_1}$. Thus, if two of them belong, say, to $\widehat{G}_{\mathcal{L}_2}$, it follows that $\widehat{G}_{\mathcal{L}_1} = \widehat{G}_{\mathcal{L}_2}$ hence $\mathcal{L}_1 = \mathcal{L}_2$. So they lie in separate groups. As there are three groups to choose, each of those elements appears precisely once.

A straightforward extension of the above argument shows that there exist three elements $g_1, g_2, g_3 \in \mathbb{T}^d$, all of order 2 and with $g_1 + g_2 \neq g_3$, such that

$$\begin{aligned}\widehat{G}_{\mathcal{L}_1} &= \{1, g_1, g_2, g_1+g_2\}\\ \widehat{G}_{\mathcal{L}_2} &= \{1, g_1, g_3, g_1+g_3\}\\ \widehat{G}_{\mathcal{L}_3} &= \{1, g_2, g_3, g_2+g_3\}\\ \widehat{G}_{\mathcal{L}_4} &= \{1, g_1+g_2, g_1+g_3, g_2+g_3\}.\end{aligned} \tag{7.4}$$

It is now straightforward to construct an automorphism $A^*\colon \mathbb{T}^d \to \mathbb{T}^d$ which maps g_1 to $(-1, 1, \ldots, 1)$, g_2 to $(1, -1, 1, \ldots, 1)$ and g_3 to $(1, 1, -1, 1, \ldots, 1)$. The dual map $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$ maps then the lattice $\mathcal{L}_1$ to $2\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{Z}^{d-2}$, $\mathcal{L}_2$ to $2\mathbb{Z}\times\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{Z}^{d-3}$, $\mathcal{L}_3$ to $\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{Z}^{d-3}$ and $\mathcal{L}_4$ to $(2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z} + ((1,1,1) + 2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z}))\times\mathbb{Z}^{d-3}$. □

8. Open questions

We end with some open questions which arise naturally from the results above.

Problem 8.1. For dimensions $d \geq 2$, give a necessary and sufficient condition, in terms of the arithmetic of the lattice translates, for a lattice tiling to possess the translation property.

We call a lattice tiling *primitive* if it is not a split tiling. In other words, a primitive lattice tiling is a tiling that we cannot form by splitting any previously formed tiling which had a smaller number of lattice translates.

Problem 8.2. Given any positive integer n, is there always a primitive lattice tiling with exactly n lattice translates?

And, of course, perhaps the most surprising state of affairs is that Question 1.3 from the introduction, concerning two dimensional lattice tilings, remains open.

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