BILIPSCHITZ MAPS AND LOGARITHMIC SPIRALS

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joint work with

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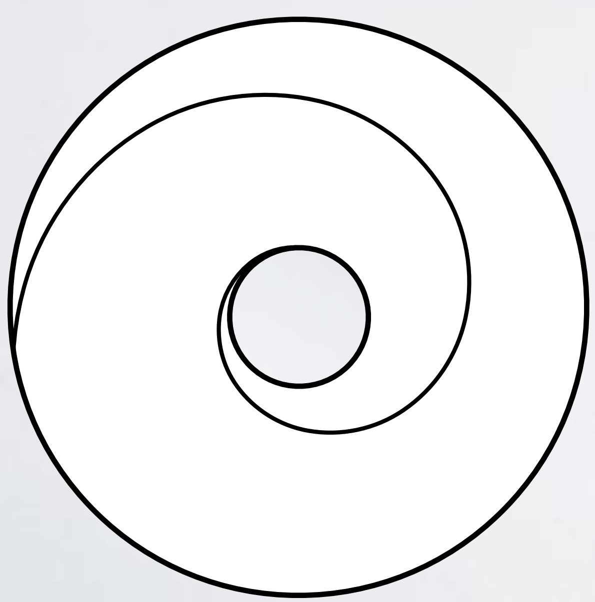
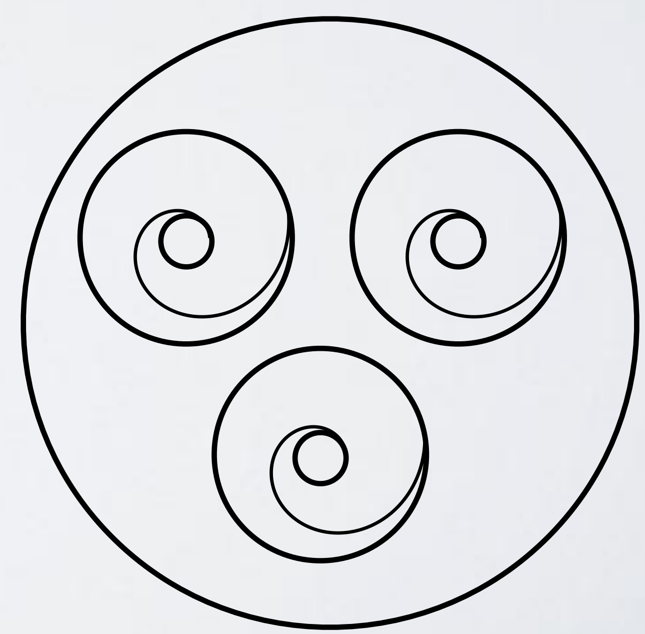
How much a bilipschitz map can spiral ?

how fast? ⟷ how often?





BILIPSCHITZ MAPS

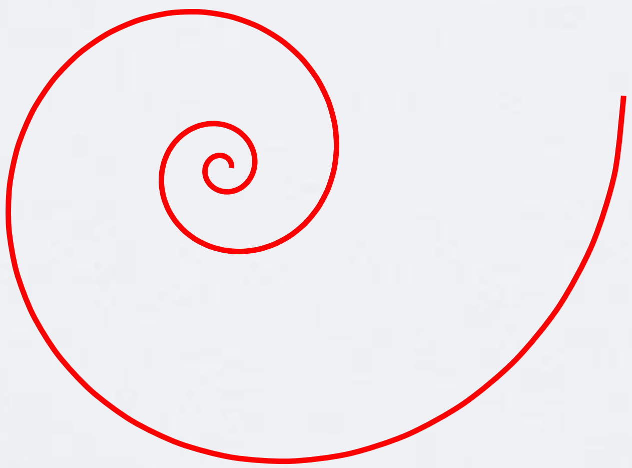
$$f\colon \Omega \to \Omega', \quad \Omega \subset \mathbb{C} \qquad L \geqslant 1$$

$$\frac{1}{L}\,|z-w| \leqslant |f(z)-f(w)| \leqslant L\,|z-w|$$

no stretching

$$\alpha(z_0) = \lim_{z\to z_0} \frac{\log|f(z)-f(z_0)|}{\log|z-z_0|} \equiv 1$$

$$\dim_H(f(E)) = \dim_H(E)$$



spiralling possible

$$g\colon \mathbb{D} \to \mathbb{D}$$

$$g(z) = z\,|z|^{i\gamma}$$

g is L-bilipschitz & area-preserving

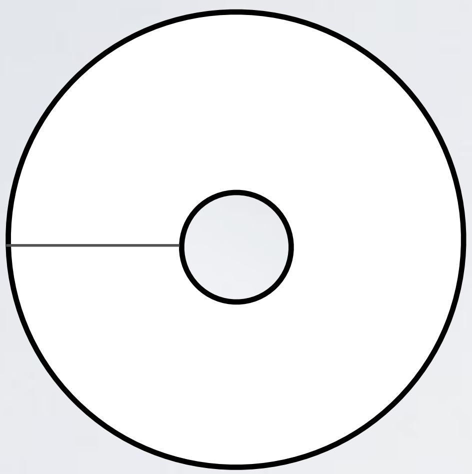
$$|\gamma| = L - \frac{1}{L}$$

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIV, 391–413 (1961)

K. O. Friedrichs anniversary issue

Rotation and Strain*

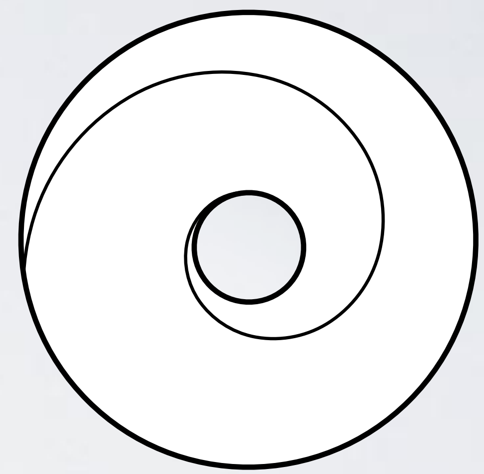
FRITZ JOHN



$$f(z) = z, \quad |z| = 1$$

$$\longrightarrow$$

L-bilipschitz



$$f(z) = z e^{i\gamma \log(r)}, \quad |z| = r$$

John: $|\gamma| \le C\varepsilon, \quad L = 1 + \varepsilon$

“stability” + BMO

John's problem

"how fast?"

John: $|\gamma| \leq C\varepsilon, \quad L = 1 + \varepsilon$

Freedman-He: $|\gamma| \leq \sqrt{L^2 - 1}$

Gutylanskii-Martio: $|\gamma| \leq L - \frac{1}{L}$ $\qquad z\,|z|^{i\gamma}$

quasiconformal techniques !

John's problem II

"how often?"

John

f is 1+ε -bilipschitz

$$\exp(b\,|\arg f_z|) \in L^1_{loc}, \quad b \sim \frac{c}{\varepsilon}$$

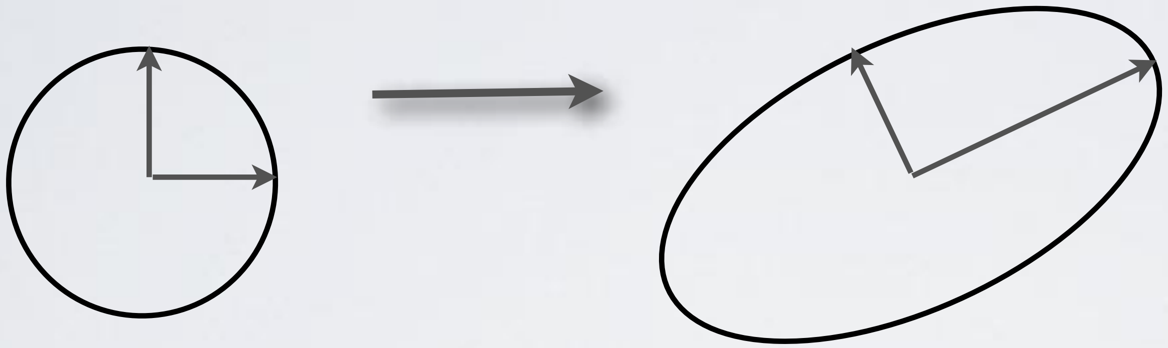
What's the best exponent *b* ?

logarithmic spiral: fails for $b \geqslant \dfrac{2L}{L^2 - 1}$

QUASICONFORMAL MAPS

$f\colon \Omega \to \Omega'$ $\quad W^{1,2}_{loc}$ — homeomorphism

$$|Df(z)|^2 \leqslant K\, J(z,f) \quad \text{a.e.} \quad z \in \Omega$$



eccentricity $\leq K$

measurable Riemann mapping theorem:
(Morrey, Vekua, Bojarski, Ahlfors-Bers,...)

can prescribe the ellipse field

BEURLING TRANSFORM

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2} dm(\zeta)$$

$$\widehat{Sf(\xi)} = \frac{\bar{\xi}^2}{|\xi|^2}\hat{f}(\xi)$$

L²-isometry

$$S \circ \partial_{\bar{z}} = \partial_z$$

Calderón-Zygmund

$$S\colon L^p \to L^p, 1 < p < \infty$$

Riesz-Thorin

$$\|S\|_p \to 1, \quad p \to 2$$

BELTRAMI EQUATION

$$f_{\bar{z}} = \mu(z)\, f_z, \quad |\mu(z)| \leqslant k\, \chi_{\mathbb{D}}(z), \quad 0 \leqslant k < 1$$

$f(z) = z + \mathcal{O}(1/z)$ principal *quasiconformal mapping*

$f \in W^{1,2}_{loc}(\mathbb{C})$ $J(z,f) > 0$ a.e. $K = \dfrac{1+k}{1-k}$

Bojarski

$$f_{\bar{z}} = (Id - \mu S)^{-1}(\mu) = \mu + \mu S\mu + \mu S(\mu S\mu) + \cdots$$

converges in L^2

$$f(z) = z + \mathcal{C}(f_{\bar{z}})(z)$$

in fact $f \in W^{1,p}_{loc}, \quad 2 \leqslant p < p_0(K)$ $\|S\|_p \overset{?}{=} p - 1$

Higher integrability

K-quasiconformal

$$f \in W^{1,p}_{loc}, \quad p < \frac{2K}{K-1}$$

Astala

holomorphic motions

+

thermodynamic formalism

L-bilipschitz

$$\exp(b\,|\arg f_z|) \in L^1_{loc},$$

$$b < \frac{2L}{L^2-1}$$

Astala-Iwaniec-Prause-Saksman

analytic dependence

+

interpolation

Quasiconformal vs Bilipschitz maps

$$f\colon \Omega \to \Omega', \quad \Omega \subset \mathbb{C}$$

flexible conformal

$$|Df(z)|^2 \leq K\, J(z,f)$$

flexible isometry

$$\frac{1}{L}\,|z-w| \leq |f(z)-f(w)| \leq L\,|z-w|$$

$$K \leq L^2$$

stretching exponent

$$\alpha(z_0) = \lim_{|z-z_0|=r_n\to 0} \frac{\log|f(z)-f(z_0)|}{\log|z-z_0|}$$

$$\alpha > 0$$

$$\frac{1}{K} \leq \alpha \leq K$$

rate of rotation

$$\gamma(z_0) = \lim_{r_n\to 0} \frac{\arg(f(z_0+r_n)-f(z_0))}{\log|f(z_0+r_n)-f(z_0)|}$$

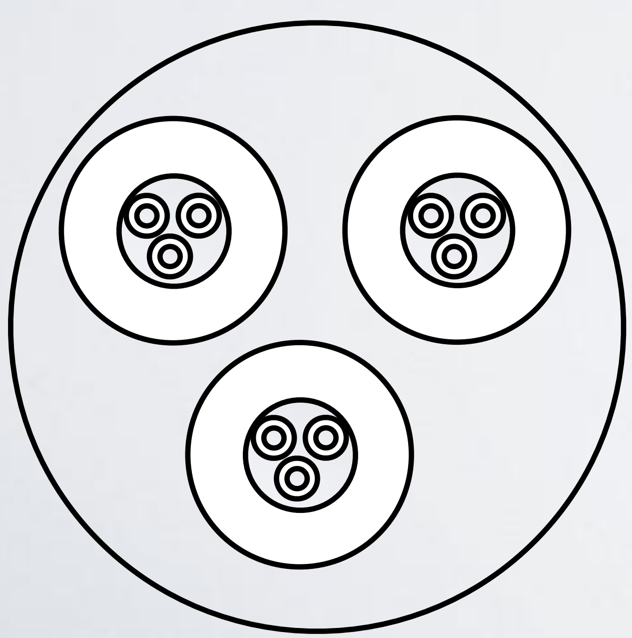
$$\gamma \in \mathbb{R}$$

$$|\gamma| \leq L - \frac{1}{L}$$

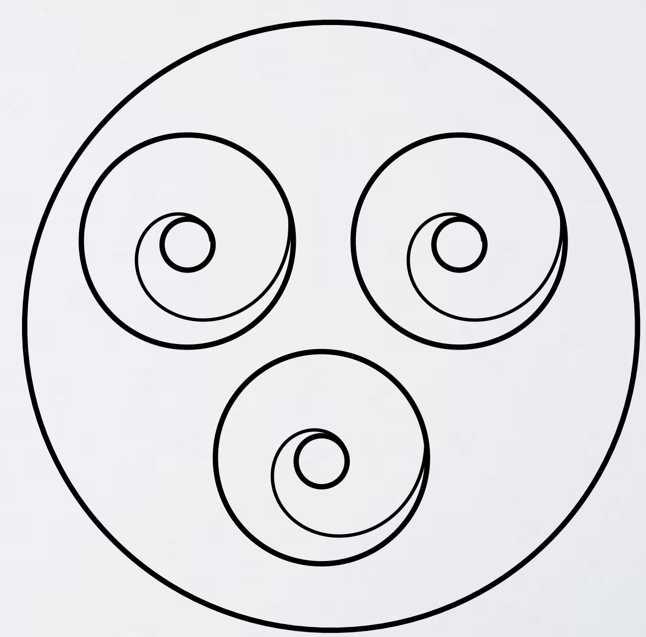
Multifractal spectra

Astala-Iwaniec-Prause-Saksman

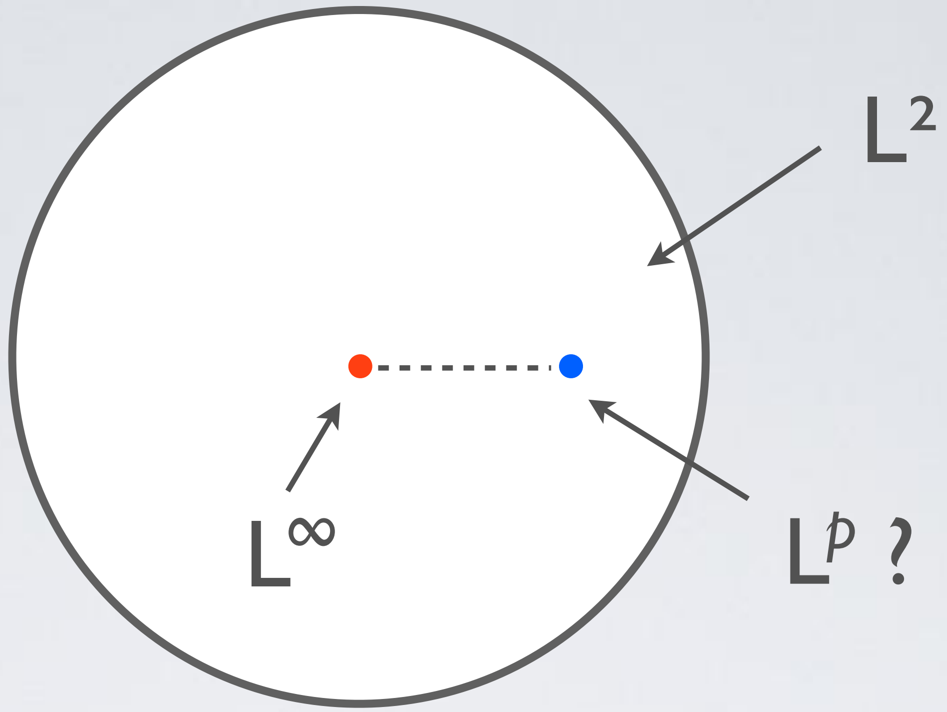
$$\dim_H\{z : \gamma(z) = \gamma\} \leq 2 - \frac{2L}{L^2 - 1}\,|\gamma|$$



L-bilipschitz



Holomorphic dependence



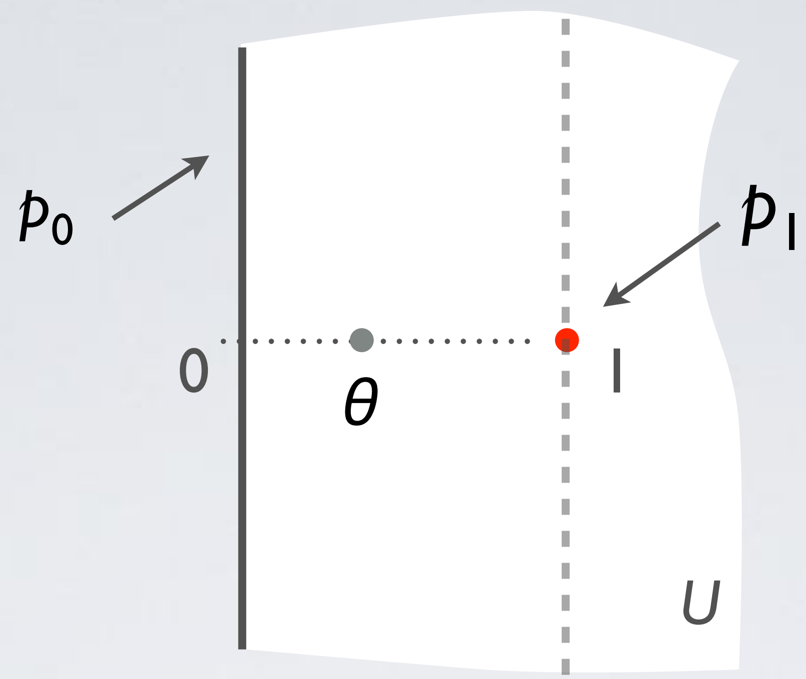
Cauchy-Riemann Beltrami

$$f^{\lambda}_{\bar{z}} = \lambda \, \mu \, f^{\lambda}_{z} \qquad \lambda \in \mathbb{D}$$

$\lambda \mapsto f^{\lambda}(z)$ holomorphic

$\lambda \mapsto f^{\lambda}_{z}(z) \neq 0$ holomorphic, for a.e. z

Interpolation Lemma



$$0 < p_0, p_1 \leqslant \infty, \quad \theta \in (0, 1)$$

$\phi_\lambda(z)$ analytic family, $\lambda \in U = \{\operatorname{Re}\lambda > 0\}$

non-vanishing $\phi_\lambda(z) \neq 0$

$$\|\phi_\lambda\|_{p_0} \leqslant M_0$$

$$\|\phi_1\|_{p_1} \leqslant M_1$$

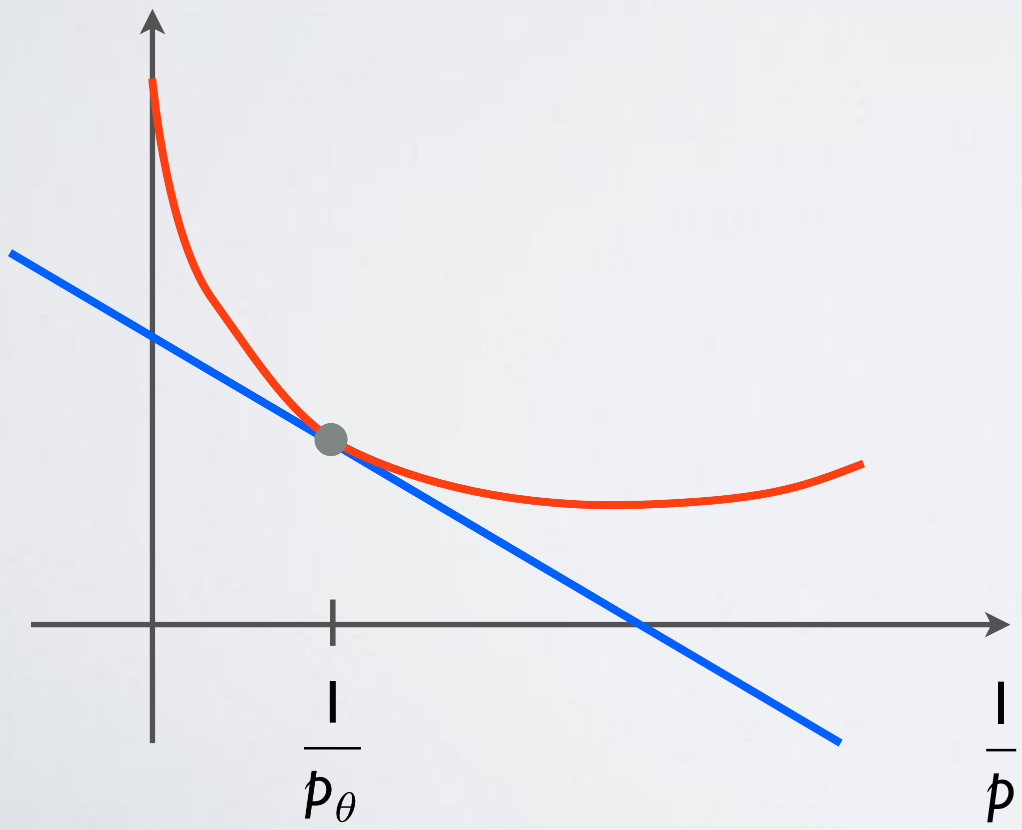
$$\Rightarrow \quad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{1-\theta} \cdot M_1^\theta$$

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

cf. Riesz-Thorin

log-convexity

change p	freeze p
subharmonic	harmonic
Hadamard	Harnack



$$\log \|\phi_\theta\|_p \geqslant A \cdot \frac{1}{p} + B$$

$$\log \|\phi_\lambda\|_p \geqslant A \cdot \frac{1}{p} + B(\lambda)$$

↑ harmonic

Interpolation Lemma, *complex version*

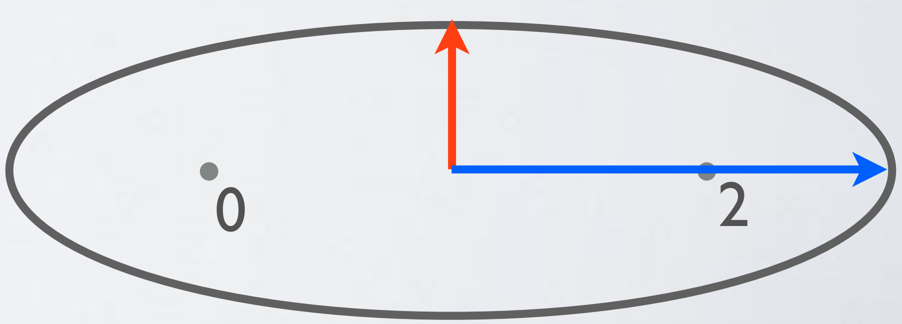
$\phi_\lambda(z), \quad \lambda \in \mathbb{D}$ *non-vanishing* analytic family,

$$\left.\begin{array}{c}\|\phi_\lambda\|_p \leqslant 1 \\ \phi_0 \equiv 1\end{array}\right. \quad \Rightarrow \quad \int \left|\phi_\lambda^\beta\right| \leq 1, \quad |\beta| + |\beta - p| \leq \frac{p}{|\lambda|}$$

harmonic ⤳ holomorphic

Harnack ⤳ Schwarz lemma

$$\phi_\lambda(z) = f_z^\lambda(z), \quad p = 2$$



foci = "null-Lagrangians"

eccentricity = ellipticity coefficient = k