On the (Fourier analytic) Sidon constant of {0,1,2,3}

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Abstract

We study an elementary extremal problem on trigonometric polynomials of degree 3. We discover a distinguished torus of extremal functions.

1 Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$ be a set of n frequencies and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We study the following extremal problem:

(‡) To find n complex coefficients $c_0, c_1, \dots, c_{n-1}$ with given moduli sum $|c_0| + |c_1| + \cdots + |c_{n-1}| = 1$ such that the maximum $\max_{z\in\mathbb{T}} |\sum c_j z^{\lambda_j}|$ is minimal.

Note that this maximum's inverse is the *Sidon constant* $S(\Lambda)$. D. J. Newman (see [4, Chapter 3]) obtained the upper bound $S(\{0, 1, \dots, N\}) \le \sqrt{N}$ that is slightly better than the straightforward upper bound $\sqrt{N+1}$: by Parseval's theorem for the L^2 space on the set $\mathbb{U}_N$ of Nth roots of unity, putting

$$c_0 + c_1 z + \cdots + c_{N-1} z^{N-1} + c_N z^N = f(z),$$

we have

$$\begin{aligned}
\max_{z\in\mathbb{T}} |f(z)|^2 &= \max_{z\in\mathbb{T}} \max_{\omega\in\mathbb{U}_N} |f(z\omega)|^2 \\
&\ge \max_{z\in\mathbb{T}} \frac{1}{N} \sum_{\omega\in\mathbb{U}_N} |f(z\omega)|^2 \\
&= \max_{z\in\mathbb{T}} |c_0 + c_N z^N|^2 + |c_1|^2 + \cdots + |c_{N-1}|^2 \\
&= \big(|c_0| + |c_N|\big)^2 + |c_1|^2 + |c_2|^2 + \cdots + |c_{N-1}|^2 \qquad (1) \\
&\ge \big(|c_0| + |c_1| + \cdots + |c_N|\big)^2/N,
\end{aligned}$$

and H. S. Shapiro showed (ibid.) that equality can hold exactly if $N \in \{1, 2, 4\}$. If $N = 3$, we shall show in the final section that the functions

$$\frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 - 3\sin\tau}{15} + \frac{3+\sin\tau}{10} z + \frac{3-\sin\tau}{10} z^2 + \frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 + 3\sin\tau}{15} z^3$$

have their modulus bounded by $3/5$ for each τ, so that $5/3 \le S(\{0, 1, 2, 3\}) < \sqrt{3}$.

A motivation for this problem is that we wish to know whether the real and complex unconditionality constants are distinct for basic sequences of characters z^n, but this remains undecided.

2 Hervé Queffélec's proof

When I showed Hervé Queffélec a proof that $S(\Lambda) \le \sqrt{n-1}$ for all sets Λ with n elements, he showed me how to adapt D. J. Newman's argument to this more general case.

Let Λ be a set of n frequencies. We may suppose that $\min\Lambda = 0 = \lambda_0$; let $N = \max\Lambda = \lambda_{n-1}$. Then

$$\begin{aligned}
\max_{z\in\mathbb{T}} |f(z)|^2 &= \max_{z\in\mathbb{T}} \max_{\omega\in\mathbb{U}_N} |f(z\omega)|^2 \\
&\geq \max_{z\in\mathbb{T}} \frac{1}{N} \sum_{\omega\in\mathbb{U}_N} |f(z\omega)|^2 \\
&= \max_{z\in\mathbb{T}} |c_0 + c_{n-1} z^{\lambda_{n-1}}|^2 + |c_1|^2 + \cdots + |c_{n-2}|^2 \\
&= \bigl(|c_0| + |c_{n-1}|\bigr)^2 + |c_1|^2 + |c_2|^2 + \cdots + |c_{n-2}|^2 \\
&\geq \bigl(|c_0| + |c_1| + \cdots + |c_{n-1}|\bigr)^2/(n-1).
\end{aligned}$$

Let us now try to understand what is behind D. J. Newman's argument.

3 Interpolating linear functionals on the space $\mathrm{C}_\Lambda(\mathbb{T})$

If L is a subspace of the space $\mathrm{C}(T)$ of complex continuous functions on a compact space T with n dimensions, then every functional l on L extends isometrically to a functional on $\mathrm{C}(T)$ by the Hahn-Banach theorem, that is, to a Radon measure by the Riesz representation theorem. But the unit ball of the space of measures is the weak*-closed convex hull of Dirac masses. By Carathéodory's theorem for the space L that has $2n$ real dimensions, l extends isometrically to a linear combination of at most $2n+1$ Dirac masses. Under additional hypotheses that are met in our situation where $T = \mathbb{T}$, one can gain one dimension: there are $m \leq 2n$ points $z_k \in T$ and coefficients $b_k \in \mathbb{C}$ such that for every $f \in L$ one has $l(f) = \sum b_k f(z_k)$ and $\|l\| = \sum |b_k|$ (see [2, Exercice 6.8].) This implies in particular that there is a function $f \in L$ whose maximum modulus points contain the z_k.

Let us now specialise to the case $L = \mathrm{C}_\Lambda(\mathbb{T})$ with Λ a finite set. Let us make the ad hoc hypothesis that the z_k are the Nth roots of unity, whose set forms the group $\mathbb{U}_N$: this obliges us to restrict our study to those functionals l such that $l(\mathrm{e}_j) = l(\mathrm{e}_{j'})$ if $j \equiv j' \bmod N$, where we write $\mathrm{e}_j(z) = z^j$ for $z \in \mathbb{T}$ and $j \in \mathbb{Z}$. Then the condition $l(f) = \sum b_k f(z_k)$ reads

$$l(\mathrm{e}_j) = \sum_{k=0}^{N-1} b_k \mathrm{e}^{\mathrm{i}2jk\pi/N} \text{ for } j \in \Lambda,$$

which may be interpreted as telling that the $l(\mathrm{e}_j)$ are the Fourier coefficients of the measure μ on $\mathbb{U}_N$ given by

$$\mu = \sum_{k=0}^{N-1} b_k \delta_{\mathrm{e}^{\mathrm{i}2k\pi/N}}$$

(where the Dirac measures act on $\mathbb{U}_N$). The set Λ might not be present in all classes modulo N: let us set $l(\mathrm{e}_j) = 0$ if j is in a class in which Λ is absent. A "trivial" solution to these equations is then given by

$$b_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}2jk\pi/N} \begin{cases} l(\mathrm{e}_{j'}) & \text{if there is } j' \equiv j \bmod N \text{ in } \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

The norm of μ is bounded by

$$\sum_{k=0}^{N-1} |b_k|$$

and is attained at $u \in \mathrm{C}(\mathbb{U}_N)$ if and only if $u(\mathrm{e}^{\mathrm{i}2k\pi/N}) b_k = |b_k|$ for every k, up to a nonzero complex factor. This yields an upper bound for the norm of l that becomes an equality if there is an $f \in \mathrm{C}(\mathbb{T})$ of norm 1 such that $f(\mathrm{e}^{\mathrm{i}2k\pi/N}) = u(\mathrm{e}^{\mathrm{i}2k\pi/N})$.

4 My proof

Here is a first application. The Sidon constant of a set Λ is also the supremum of the norm of the linear functionals l such that $l(e_j)$ is a unimodular complex number for all $j \in \Lambda$:

$$|c_0| + |c_1| + \cdots + |c_{n-1}| = \sup_{|l(\mathrm{e}_j)|=1} \left| \sum_{j=0}^{n-1} c_j l(\mathrm{e}_j) \right|.$$

Proposition 4.1. *Let Λ be a finite subset of $\mathbb{Z}$. The Sidon constant of Λ is at most $(\#\,\Lambda-1)^{1/2}$.*

Proof. One may suppose that $\min\Lambda = 0$ and choose $N = \max\Lambda$. Let l be a linear functional with coefficients $l(\mathrm{e}_j)$ of modulus 1: one may suppose that $l(\mathrm{e}_0) = l(\mathrm{e}_N)$. Then

$$\begin{aligned} \|l\| &\le \frac{1}{N} \sum_{k=0}^{N-1} \left| \sum_{j\in\Lambda\setminus\{N\}} \mathrm{e}^{-\mathrm{i}2jk\pi/N} l(\mathrm{e}_j) \right| \\ &\le \left(\frac{1}{N} \sum_{k=0}^{N-1} \left| \sum_{j\in\Lambda\setminus\{N\}} \mathrm{e}^{-\mathrm{i}2jk\pi/N} l(\mathrm{e}_j) \right|^2 \right)^{1/2} \qquad (2) \\ &= \left(\sum_{j\in\Lambda\setminus\{N\}} |l(\mathrm{e}_j)|^2 \right)^{1/2} = (\#\,\Lambda - 1)^{1/2}. \end{aligned}$$

□

Remark 4.2. If $\Lambda = \{0, 1, \ldots, n\}$, then Inequality (2) is an equality if and only if $(l(\mathrm{e}_j))_{j=0}^{n-1}$ is a *biunimodular* sequence, that is a unimodular function on $\mathbb{U}_n$ whose Fourier transform is also unimodular. In other words, the matrix $H = (l(\mathrm{e}_{j-k}))_{0\le j,k\le n-1}$ is a *circulant complex Hadamard matrix*, where the indices $j - k$ are computed modulo n: it satisfies $H^*H = n\,\mathrm{Id}$. Such matrices always exist: see [1].

5 The real unconditional constant of $\{0, 1, 2, 3\}$

Here is a second application. Recall that the real unconditional constant of a sequence of elements of a normed space is the maximal distortion caused by multiplying the coefficients of a linear combination of these elements by ±1. By a slight abuse of language, the real unconditional constant of a set Λ in the space $\mathrm{C}(\mathbb{T})$ is thus the supremum of the norm of the linear functionals l such that $l(\mathrm{e}_j) \in \{-1, 1\}$ for all $j \in \Lambda$.

Proposition 5.1. *Let $\Lambda = \{0, 1, 2, 3\}$. The real unconditional constant of Λ in $\mathrm{C}(\mathbb{T})$ is $5/3$.*

Proof. The polynomial $-4/15+2z/5+z^2/5+2z^3/15$ studied in the next section will show that the real unconditional constant of $\mathrm{C}_\Lambda(\mathbb{T})$ is at least $5/3$. As l has the same norm as $\tilde{l}\colon f \mapsto l(f(\cdot+\pi))$, for which $\tilde{l}(\mathrm{e}_j) = (-1)^j l(\mathrm{e}_j)$, and as $-l$, one may suppose that $l(\mathrm{e}_0) = l(\mathrm{e}_3) = 1$. Let us now try to lift l to a sum of Dirac measures on the third roots of unity. Such a lifting is either the Dirac measure at 0 or

$$(l(\mathrm{e}_j)_{0\le j\le 2}) \in \{(1, -1, -1), (1, -1, 1), (1, 1, -1)\}$$

and these three cases yield the same norm

$$\frac{1}{3}(|1 - 1 - 1| + |1 - \mathrm{e}^{\mathrm{i}2\pi/3} - \mathrm{e}^{\mathrm{i}4\pi/3}| + |1 - \mathrm{e}^{\mathrm{i}4\pi/3} - \mathrm{e}^{\mathrm{i}2\pi/3}|) = 5/3.$$

□

6 The case $\{0, 1, 2, 3\}$: a distinguished family of polynomials

Let $f(z, \tau)$ be given by

$$\frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 - 3\sin\tau}{15} + \frac{3 + \sin\tau}{10} z + \frac{3 - \sin\tau}{10} z^2 + \frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 + 3\sin\tau}{15} z^3.$$

One computes that the moduli sum of the coefficients is 1, independently of τ. Note that $f(z,-\tau) = z^3 f(z^{-1},\tau)$ and $f(z,\tau+\pi) = z^3\overline{f(z,\tau)}$, so that we shall restrict the parameter τ to $[0,\pi/2]$. Let $\Phi(t,\tau) = |f(\mathrm{e}^{\mathrm{i}t},\tau)|^2$. We get

$$\Phi(t,\tau) = \frac{2\sqrt{2}\sin 2\tau}{75}(\sin t - \sin 2t + 2\sin 3t) + \frac{247 - 13\cos 2\tau}{900} + (1+\cos 2\tau)\Big(\frac{\cos t}{20} - \frac{\cos 2t}{25}\Big) + \frac{1 + 17\cos 2\tau}{225}\cos 3t.$$

Let us put

$$M = \begin{pmatrix} \frac{2\sin 2t}{25} - \frac{\sin t}{20} - \frac{17\sin 3t}{75} & \frac{2\sqrt{2}}{75}(\cos t - 2\cos 2t + 6\cos 3t) \\ \frac{2\sqrt{2}}{75}(\sin t - \sin 2t + 2\sin 3t) & \frac{13}{900} - \frac{\cos t}{20} + \frac{\cos 2t}{25} - \frac{17\cos 3t}{225} \end{pmatrix}.$$

The critical points (t,τ) of Φ satisfy

$$M\begin{pmatrix}\cos 2\tau \\ \sin 2\tau\end{pmatrix} = \begin{pmatrix} \frac{\sin t}{20} - \frac{2\sin 2t}{25} + \frac{\sin 3t}{75} \\ 0 \end{pmatrix}.$$

We have

$$\det M = \frac{1}{6750}\sin t\Big(\cos t - \frac{1}{4}\Big)(4\cos t - 11)(16\cos^3 t - 72\cos^2 t + 33\cos t - 41),$$

which vanishes exactly if $\cos t \in \{-1, 1/4, 1\}$. Otherwise we get

$$\begin{cases} \cos 2\tau = -\dfrac{272\cos^3 t - 72\cos^2 t - 159\cos t + 23}{16\cos^3 t - 72\cos^2 t + 33\cos t - 41} = C(t) \\ \sin 2\tau = -\dfrac{24\sqrt{2}\sin t(4\cos t + 1)(2\cos t - 1)}{16\cos^3 t - 72\cos^2 t + 33\cos t - 41} = S(t) \end{cases} \tag{3}$$

Note that this solution is consistent, as $C^2 + S^2 = 1$. For such τ, $\Phi(t,\tau) = 9/25$. Checking the special cases $\cos t \in \{-1, 1/4, 1\}$ yields that all local maxima are given by the above formulas, that Φ attains its global minimum, 0, exactly for $\tau = 0$ and $t = \pi$, and has exactly one other local minimum, of value $49/225$, for $\tau = \pi/2$ and $t = 0$. There is exactly one other critical point, of value $5/18$, that is a saddle point, given by $\tau = \arccos(17/37)/2$, $t = \arccos 1/4$.

As $C(0) = 1$, $C(\pm\pi/3) = -1$, $C(\pm\arccos(-1/4)) = 1$, $C(\pi) = -1$, the intermediate values theorem shows that for a given τ, there are exactly three solutions t to system (3), for which $\Phi(t,\tau)$ achieves then its global maximum, $9/25$.

Further details are given in [3].

References

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