Statistical convergence of sequences and series of complex numbers with applications in Fourier Analysis and Summability

FERENC MÓRICZ

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1,

6720 Szeged, Hungary

e-mail: moricz@math.u-szeged.hu

Abstract. This is a survey paper on recent results indicated in the title. In contrast to the famous examples of Kolmogorov and Fejér on the pointwise divergence of Fourier series, the statistical convergence of the Fourier series of any integrable function takes place at almost every point; and the statistical convergencr of the Fourier series of any continuous function is uniform. Furthermore, Tauberian conditions are also presented, under which ordinary convergence of any sequence of real or complex numbers follows from its statistical summability.

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1. Statistical convergence of single sequences and series

The term 'statistical convergence' first appeared in 1951, see [2] by Fast, where he attributed this concept to Hugo Steinhaus. In fact, it was Zygmund who first proved theorems on the statistical convergence of Fourier series in the first edition of his book "Trigonometric Series" appeared in 1935, where the term 'almost convergence' was used in place of 'statistical convergence'.

We recall that a sequence $(s_k) = (s_k : k = 0, 1, 2, \ldots)$ or a series with partial sums s_k, where the s_k are real or complex numbers, is said to *converge statistically* to limit (or sum) s, in symbols:

$$\text{st} - \lim_{k\to\infty} s_k = s, \tag{1.1}$$

if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} (n+1)^{-1} |\{k \le n : |s_k - s| > \varepsilon\}| = 0, \tag{1.2}$$

where by $k \le n$ we mean that $k = 0, 1, 2, \ldots, n$; and by $|\mathcal{S}|$ we mean the cardinality of the finite set $\mathcal{S} \subset \mathbb{N} := \{0, 1, 2, \ldots\}$.

It is clear that the statistical limit s in (1.1) is uniquely determined if it exists. The concept of statistical limit also enjoys all the limit laws that are valid in the case of the ordinary limit in the sense of Cauchy: additivity, homogeneity, etc. Clearly, the existence of the ordinary limit of a sequence implies the existence of the statistical limit, and the two limits coincide. The converse implication is not true in general. For example, if the sequence (s_k) is defined by

$$s_k := \begin{cases} k & \text{if} \quad k = 2^\ell, \ \ell = 0, 1, 2, \ldots, \\ 0 & \text{otherwise}; \end{cases}$$

then (1.2) with $s = 0$ is satisfied for every $\varepsilon > 0$, while the sequence (s_k) is not bounded.

We recall that a sequence (s_k) is said to be *statistically bounded* if there exists some constant $B > 0$ such that

$$\lim_{n\to\infty} (n+1)^{-1} |\{k \le n : |s_k| > B\}| = 0.$$

If (1.2) holds for some $\varepsilon > 0$, then this inequality clearly holds with $B := |s| + \varepsilon$. Consequently, every statistically convergent sequence is statistically bounded.

The following concept was introduced by Fridy [3] in 1985. A sequence (s_k) is said to be *statistically Cauchy* if for every $\varepsilon > 0$ there exists $\nu = \nu(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{n\to\infty} (n+1)^{-1} |\{k \le n : |s_k - s_\nu| > \varepsilon\}| = 0.$$

Theorem 1.1 (Fridy [3]). *A sequence (s_k) converges statistically if and only if it is statistically Cauchy.*

We recall (see, e.g., [17, p. 290]) that the *natural* (or asymptotic) *density* of a set $\mathcal{S} \subset \mathbb{N}$ is defined by

$$d(\mathcal{S}) := \lim_{n\to\infty} (n+1)^{-1} |\{k \le n : k \in \mathcal{S}\}|,$$

provided that this limit exists. Using this term, (1.2) can be equivalently rewritten as follows: for every $\varepsilon > 0$,

$$d(\{k \in \mathbb{N} : |s_k - s| > \varepsilon\}) = 0.$$

The following *decomposition theorem* was proved by Connor [1].

Theorem 1.2. *A sequence (s_k) converges statistically to limit s if and only if there exist two sequences (u_k) and (v_k) such that*

(i) $s_k = u_k + v_k, \quad k = 0, 1, 2, \ldots,$

(ii) $\lim_{k\to\infty} u_k = s,$

(ii) $\lim_{n\to\infty} (n+1)^{-1} |\{k \le n : v_k \ne 0\}| = 0.$

Moreover, if (s_k) is bounded, then both (u_k) and (v_k) are also bounded.

According to Zygmund's definition (see [20, Vol. II, p. [181], a sequence (s_k) or a series with partial sums s_k is said to be '*almost convergent*' to limit (sum) s if there exists a sequence $0 \le k_1 < k_2 < \ldots < k_n < \ldots$ of integers such that

$$\lim_{n\to\infty} \frac{n}{k_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} s_{k_n} = s. \tag{1.3}$$

It is routine to check that the definition of statistical convergence of a sequence (s_k) defined in (1.2) is equivalent to the definition (1.3) of almost convergence of (s_k).

Furthermore, a sequence (s_k) or a series with partial sums s_k is said to be *summable H_q* to limit (or sum) s for some real $q > 0$ if

$$\lim_{n\to\infty} (n+1)^{-1} \sum_{k=0}^{n} |s_k - s|^q = 0 \tag{1.4}$$

(see in [20, Vol. II, p. 180]). We note that this kind of summability, also called *strong summability*, was first considered by Hardy and Littlewood [7].

Hölder's inequality gives that if (1.4) holds for some $q > 0$, then it also holds for every smaller exponent r, $0 < r < q$. Clearly, summability H_1, also known as strong summability $(C, 1)$, implies summability by the first arithmetic means; that is, we have

$$\lim_{n\to\infty} (n+1)^{-1} \sum_{k=0}^{n} s_k = s.$$

Theorem 1.3 (Zygmund [20, Vol. II, p. 181]). (i) *If a sequence (s_k) is summable H_q to limit s for some $q > 0$, then (s_k) is almost convergent to s.*

(ii) *Conversely, if (s_k) is almost convergent to s, and it is bounded, then for any $q > 0$, (s_k) is summable H_q to s.*

2. Application to single Fourier series

Let $f : \mathbb{T} \to \mathbb{C}$ be an integrable function in Lebesgue's sense on the torus $\mathbb{T} := [-\pi, \pi)$, in symbols: $f \in L^1(\mathbb{T})$. As is well known, the Fourier series of f is defined by

$$f(x) \sim \sum_{j\in\mathbb{Z}} \hat{f}(j)e^{ijx}, \quad x \in \mathbb{T}, \tag{2.1}$$

where the Fourier coefficients $\hat{f}(j)$ are defined by

(2.2)
$$\hat{f}(j) := \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ijt}dt, \quad j \in \mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The *symmetric partial sums* of the series in (2.1) are defined by

$$s_k(f;x) := \sum_{|j|\le k} \hat{f}(j)e^{ijx}, \quad k \in \mathbb{N} \quad \text{and} \quad x \in \mathbb{T}. \tag{2.3}$$

We recall (see, e.g., in [20, Vol. I, p. 49] that the *conjugate series* to the Fourier series in (2.1) is defined by

$$\sum_{j\in\mathbb{Z}} (-i \ \mathrm{sign} j)\hat{f}(j)e^{ijx}, \tag{2.4}$$

where

$$\mathrm{sign} j := \begin{cases} \frac{j}{|j|} & \text{if} \quad j \ne 0, \\ 0 & \text{if} \quad j = 0. \end{cases}$$

Clearly, it follows from (2.1) and (2.4) that

$$\sum_{j\in\mathbb{Z}} \hat{f}(j)e^{ijx} + i\sum_{j\in\mathbb{Z}}(-i \ \mathrm{sign} j)\hat{f}(j)e^{ijx} = 1 + 2\sum_{j=1}^{\infty} \hat{f}(j)e^{ijx},$$

and the power series

$$1 + 2\sum_{j=1}^{\infty} \hat{f}(j) z^j, \quad \text{where} \quad z := re^{ix},\ 0 \le r < 1,$$

is analytic on the open unit disk $|z| < 1$, due to the fact that

$$|\hat{f}(j)| \le \frac{1}{2\pi} \int_{\pi} |f(t)| dt, \quad j \in \mathbb{Z}.$$

This justifies the term 'conjugate series' in the case of (2.4).

We also recall (see, e.g., in [20, Vol. I, p. 51] that the *conjugate function* $\widetilde{f}$ of a function $f \in L^1(\mathbb{T})$ is defined by

$$\widetilde{f}(x) := -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\varepsilon \le |t| \le \pi} \frac{f(x+t)}{2\tan\frac{t}{2}} dt = \tag{2.5}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{f(x-t) - f(x+t)}{2\tan\frac{t}{2}} dt$$

in the 'principal value' sense; and that $\widetilde{f}(x)$ exists at almost every $x \in \mathbb{T}$.

In the following Theorem 2.1, Part (i) was proved by Hardy and Littlewood [7], while Part (ii) is due to Marcinkiewicz [10, for $q = 2$] and Zygmund [19, for every $q > 0$].

Theorem 2.1. (i) *If a periodic function f is continuous, in symbols: $f \in C(\mathbb{T})$, then for any $q > 0$ its Fourier series is summable H_q to $f(x)$ uniformly in $x \in \mathbb{T}$.*

(ii) *If* $f \in L^1(\mathbb{T})$, *then for any* $q > 0$ *its Fourier series is summable* H_q *to* $f(x)$ *at almost every* $x \in \mathbb{T}$. *Furthermore, its conjugate series* (2.4) *is summable* H_q *for any* $q > 0$ *to the conjugate function* $\widetilde{f}(x)$ *defined in* (2.5) *at almost every* $x \in \mathbb{T}$.

The next Corollary 2.2. follows from Theorems 1.3 and 2.1.

Corollary 2.2. (i) *If* $f \in C(\mathbb{T})$, *then its Fourier series statistically converges to* $f(x)$ *uniformly in* $x \in \mathbb{T}$.

(ii) *If* $f \in L^1(\mathbb{T})$, *then its Fourier series statistically converges to* $f(x)$ *at almost every* $x \in \mathbb{T}$. *Furthermore, its conjugate series statistically converges to the conjugate function* $\widetilde{f}(x)$ *at almost every* $x \in \mathbb{T}$.

By virtue of Corollary 2.2, the ordinary divergence (in the sense of Cauchy) of the Fourier series of a function $f \in C(\mathbb{T})$ at a single point $x \in \mathbb{T}$ (see the example in [20, Vol. I, p. 299] given by L. Fejér) is due to the existence of a subsequence $0 < \ell_1 < \ell_2 < \ldots < \ell_p < \ldots$ of integers with natural density 0 such that the subsequence $(s_{\ell_p}(f;x))$ of the partial sums of the Fourier series of f diverges at the point x in question. An analogous observation can be also made in the case when the Fourier series of a function $f \in L^1(\mathbb{T})$ diverges everywhere (see the example in [20, Vol. I, p. 310] given by A. N. Kolmogorov).

3. Statistical convergence of multiple sequences and series

The concepts and results of the preceeding two Sections can be extended to m-multiple sequences and series, where $m \geq 2$ is a fixed integer. Denote by $\mathbb{N}^m$ the set of m-tuples $\boldsymbol{k} = (k_1, k_2, \ldots, k_m)$ with nonnegative integers for the coordinates k_j. Two tuples $\boldsymbol{k}$ and $\boldsymbol{n} = (n_1, n_2, \ldots, n_m)$ are distinct if and only if $k_\ell \neq n_\ell$ for at least one ℓ, $1 \leq \ell \leq m$. As is known, $\mathbb{N}^m$ is partially ordered by agreeing that $\boldsymbol{k} \leq \boldsymbol{n}$ if $k_\ell \leq n_\ell$ for each $\ell = 1, 2, \ldots, m$.

According to [11] (see also in [16] in the case $m = 2$), an m-multiple sequence $(s_{\boldsymbol{k}}) = (s_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}^m)$ of real or complex numbers is said to *converge statistically* to limit s, in symbols:

$$\mathrm{st} - \lim_{\boldsymbol{k} \to \infty} s_{\boldsymbol{k}} = s,$$

if for every $\varepsilon > 0$,

$$\lim_{\boldsymbol{n} \to \infty} \prod_{\ell=1}^{m} (n_\ell + 1)^{-1} |\{\boldsymbol{k} \leq \boldsymbol{n} : |s_{\boldsymbol{k}} - s| > \varepsilon\}| = 0$$

(cf. (1.2)), where by $\boldsymbol{n} \to \infty$ we mean that

$$\min\{n_1, n_2, \ldots, n_m\} \to \infty.$$

In the case of an m-multiple series

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \dots \sum_{j_m=0}^{\infty} c_{j_1,j_2,\dots,j_m} \tag{3.1}$$

of real of complex numbers, we consider its *rectangular partial sums* $s_{\boldsymbol{k}}$ defined by

$$s_{\boldsymbol{k}} := \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \dots \sum_{j_m=0}^{k_m} c_{j_1,j_2,\dots,j_m}, \quad \boldsymbol{k} \in \mathbb{N}^m.$$

The m-multiple series (3.1) is said to converge statistically to sum s if the m-multiple sequence $(s_{\boldsymbol{k}})$ of its rectangular partial sums statistically converges to s.

Furthermore, a multiple sequence $(s_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}^m)$ is said to be *statistically Cauchy* if for every $\varepsilon > 0$ there exists $\boldsymbol{\nu} \in \mathbb{N}^m$ such that

$$\lim_{\boldsymbol{n}\to\infty} \prod_{\ell=1}^{m} (n_\ell + 1)^{-1} |\{\boldsymbol{k} \le \boldsymbol{n} : |s_{\boldsymbol{k}} - s_{\boldsymbol{\nu}}| > \varepsilon\}| = 0.$$

Both Theorems 1.1 and 1.2 are valid for m-multiple sequences; see the proofs in [11] for double sequences. The *natural* (or asymptotic) *density* of a set $\mathcal{S} \subset \mathbb{N}^m$ can be defined as follows

$$d(\mathcal{S}) := \lim_{\boldsymbol{n}\to\infty} \prod_{j=1}^{m} (n_j + 1)^{-1} |\{\boldsymbol{k} \le \boldsymbol{n} : \boldsymbol{k} \in \mathcal{S}\}|,$$

provided this limit exists.

An m-multiple sequence $(s_{\boldsymbol{k}})$ or series with rectangular partial sums $s_{\boldsymbol{k}}$ is said to be summable H_q to limit (or sum) s for some real $q > 0$ if

$$\lim_{\boldsymbol{n}\to\infty} \prod_{j=1}^{m} (n_j+1)^{-1} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \dots \sum_{k_m=0}^{n_m} |s_{\boldsymbol{k}} - s|^q = 0. \tag{3.2}$$

It is easy to check that Theorem 1.3 is also valid in the case of m-multiple sequences or series.

Historical remark. The manuscript of [11] of the present author was received by the Editors on November 5, 2001; while the manuscripot of [16] of Mursaleen and Osama H.H. Edely was received by the Editors on January 11, 2002; and in the latter one only the case $m = 2$ is considered.

4. Application to m-multiple Fourier series

Let $f : \mathbb{T}^m \to \mathbb{C}$ be an integrable function in Lebesgue's sense on the m-dimensional torus $\mathbb{T}^m := [-\pi, \pi)^m$, where $m \geq 2$ is an integer. We recall (see, e.g., [20, Vol. II, Ch. 17]) that the m-multiple Fourier series of f is defined by

$$f(\boldsymbol{x}) \sim \sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \dots \sum_{j_m\in\mathbb{Z}} \hat{f}(\boldsymbol{j}) e^{i\boldsymbol{j}\cdot\boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{T}^m, \tag{4.1}$$

where the Fourier coefficients $\hat{f}(j)$ are defined (cf. (2.2)) by

$$\hat{f}(\boldsymbol{j}) := \frac{1}{(2\pi)^m} \int_{\mathbb{T}} \int_{\mathbb{T}} \dots \int_{\mathbb{T}} f(\boldsymbol{t}) e^{i\boldsymbol{j}\cdot\boldsymbol{t}} dt_1 dt_2 \dots dt_m,$$

$$\boldsymbol{j}\cdot\boldsymbol{t} := \sum_{\ell=1}^{m} j_\ell t_\ell, \quad \boldsymbol{j} \in \mathbb{Z}^m \quad \text{and} \quad \boldsymbol{t} \in \mathbb{T}^m.$$

The *symmetric rectangular partial sums* of the m-multiple series in (4.1) are defined (cf. (2.3)) by

$$s_{\boldsymbol{k}}(f;\boldsymbol{x}) := \tag{4.2}$$

$$\sum_{|j_1|\le k_1} \sum_{|j_2|\le k_2} \dots \sum_{|j_m|\le k_m} \hat{f}(\boldsymbol{j}) e^{i\boldsymbol{j}\cdot\boldsymbol{x}}, \quad \boldsymbol{k} \in \mathbb{N}^m \quad \text{and} \quad \boldsymbol{x} \in \mathbb{T}^m.$$

The convergence of these $s_{\boldsymbol{k}}(f;\boldsymbol{x})$ is meant *in Pringsheim's sense*, that is, when the finite limit

$$\lim s_{\boldsymbol{k}}(f;\boldsymbol{x}) \quad \text{exists as} \quad \min\{k_1, k_2 \dots, k_m\} \to \infty.$$

For the sake of brevity in writing, first we define the *conjugate series* to the Fourier series in (4.1) in the case when $m = 2$ as follows (see, e.g., in [18]):

$$\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} (-i \,\operatorname{sign} j_1) \hat{f}(\boldsymbol{j}) e^{i\boldsymbol{j}\cdot\boldsymbol{x}},$$

$$\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} (-i \,\operatorname{sign} j_2) \hat{f}(\boldsymbol{j}) e^{i\boldsymbol{j}\cdot\boldsymbol{x}},$$

$$\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}}(-i\ \mathrm{sign} j_1)(-i\ \mathrm{sign} j_2)\hat{f}(\boldsymbol{j})e^{n\boldsymbol{j}\cdot\boldsymbol{x}}, \quad \boldsymbol{x}\in\mathbb{T}^2.$$

Analogously to the one-dimensional case, the corresponding *conjugate functions* $\widetilde{f}^{(1,0)}(\boldsymbol{x})$, $\widetilde{f}^{(0,1)}(\boldsymbol{x})$ and $\widetilde{f}^{(1,1)}(\boldsymbol{x})$ are defined by

$$\widetilde{f}^{|(1,0)}(\boldsymbol{x}) := -\lim_{\varepsilon_1\downarrow 0}\frac{1}{\pi}\int_{\varepsilon_1\le|t_1|\le\pi}\frac{f(x_1+t_1,x_2)}{2\tan\frac{t_1}{2}}dt_1,$$

$$\widetilde{f}^{(0,1)}(\boldsymbol{x}) := -\lim_{\varepsilon_2\downarrow 0}\frac{1}{\pi}\int_{\varepsilon_2\le|t_2|\le\pi}\frac{f(x_1,x_2+t_2)}{2\tan\frac{t_2}{2}}dt_2,$$

$$\widetilde{f}^{(1,1)}(\boldsymbol{x}) :=$$

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\frac{1}{\pi^2}\int_{\varepsilon_1\le|t_1|\le\pi}\int_{\varepsilon_2\le|t_2|\le\pi}\frac{f(x_1+t_1,x_2+t_2)}{4\tan\frac{t_1}{2}\tan\frac{t_2}{2}}dt_1dt_2,$$

each integral is meant in the '*principal value*' sense. It is also known (see, e.g., in [4] or one of the References given there), each of these conjugate functions exists at almost every $\boldsymbol{x}\in\mathbb{T}^2$ provided that

$$\int_{\mathbb{T}}\int_{\mathbb{T}}|f(t_1,t_2)|\log^+|f(t_1,t_2)|dt_1dt_2<\infty, \tag{4.3}$$

in symbols: $f\in L^1\log^+ L(\mathbb{T}^2)$, where

$$\log^+|u| := \max\{0,\log|u|\} \quad u\in\mathbb{C}.$$

In the general case, where $m \geq 3$, let

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_m) \in \{0,1\}^m$$

be such that at least one of its components equals 1. More precisely, let $\eta_{\ell_1} = \ldots = \eta_{\ell_s} = 1$,

$$\text{where} \quad 1 \leq \ell_1 < \ldots < \ell_s \leq m \quad \text{and} \quad 1 \leq s < m;$$

while $\eta_\ell = 0$ for the remaning indices ℓ (if any) between 1 and m. Then the m-multiple series

$$\sum_{j_1 \in \mathbb{Z}} \ldots \sum_{j_s \in \mathbb{Z}} (-i \ \mathrm{sign} j_{\ell_1}) \ldots (-i \ \mathrm{sign} j_{\ell_s}) \hat{f}(\boldsymbol{j}) e^{i \boldsymbol{j} \cdot \boldsymbol{x}} \tag{4.4}$$

is the *conjugate series* to the Fourier series in (4.1) that corresponds to $\boldsymbol{\eta} \in \{0,1\}^m$ indicated above. Altogether, there are $2^m - 1$ conjugate series to the Fourier series in (4.1). The symmetric rectangular partial sums of the m-multiple series (4.4) (analogously to the notation in (4.2)) are denoted by $\widetilde{s}_{\boldsymbol{k}}^{(\boldsymbol{\eta})}(f; \boldsymbol{x})$.

Analogously to the two-dimensional case, there are $2^m - 1$ conjugate functions to the function f in (4.1), which we denote by $\widetilde{f}^{(\boldsymbol{\eta})}(\boldsymbol{x})$, $\boldsymbol{\eta} \in \{0,1\}^m$ with at least one component $\eta_\ell = 1$. Each of these conjugate functions exists as a principal value integral at almost every $\boldsymbol{x} \in \mathbb{T}^m$ provided that

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} |f(t_1, t_2 \ldots, t_m)|$$

$$\times(\log^+|f(t_1,t_2\ldots,t_m)|)^{m-1}dt_1dt_2\ldots dt_m<\infty$$

(cf. (4.3)), in symbols: $f\in L^1(\log^+ L)^{m-1}(\mathbb{T}^m)$.

The analogue of Theorem 2.1 reads as follows.

Theorem 4.1 (Gogoladze [4]). *If $f\in L^1(\log^+ L)^{m-1}(\mathbb{T}^m)$, then the m-multiple sequence $(s_{\boldsymbol{k}}(f;\boldsymbol{x}))$ of the symmetric rectangular partial sums of the Fourier series in* (4.1) *is summable H_q to $f(\boldsymbol{x})$ (*see in definition (3.2)) *at almost every point $\boldsymbol{x}\in\mathbb{T}^m$ for any $0<q<\infty$.*

Furthermore, for each $\boldsymbol{\eta}\in\{0,1\}^m$ with at least one component $\eta_\ell=1$, the m-multiple sequence $(s^{(\boldsymbol{\eta})}_{\boldsymbol{k}}(f;\boldsymbol{x}))$ of the conjugate series (4.4) *is also summable H_q at almost every point $\boldsymbol{x}\in\mathbb{T}^m$ to the conjugate function $\widetilde{f}^{(\boldsymbol{\eta})}(\boldsymbol{x})$ for any $0<q<\infty$.*

The next Corollary 4.2 follows from Theorems 4.1 and the m-multiple version of Theorem 1.3.

Corollary 4.2. *If $f\in L^1(\log^+ L)^{m-1}(\mathbb{T}^m)$, then the m-multiple Fourier series in* (4.1) *statistically converges to $f(\boldsymbol{x})$ at almost every $\boldsymbol{x}\in\mathbb{T}^m$. Furthermore, each of its conjugate series* (4.4) *statistically converges to the corresponding conjugate function $\widetilde{f}^{(\boldsymbol{\eta})}(\boldsymbol{x})$ at almost every $\boldsymbol{x}\in\mathbb{T}^m$.*

In a joint paper [15] with Xianliang Shi, we proved the following

Theorem 4.3. *For any $0 < q < \infty$, there exists a constant C_q depending only on q such that if $f \in C(\mathbb{T}^2)$, then for all $(n_1, n_2) \in \mathbb{N}^2$ and $(x_1, x_2) \in \mathbb{T}^2$ we have*

(4.4)

$$(n_1+1)^{-1}(n_2+1)^{-1} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} |s_{j_1,j_2}(f; x_1, x_2) - f(x_1, x_2)|^q$$

$$\leq C_q (n_1+1)^{-1}(n_2+1)^{-1} \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} (E_{j_1,j_2}(f))^q,$$

where $E_{j_1,j_2}(f)$ is the best uniform approximation to the continuous function f by two-dimensional trigonometric polynomials $T(t_1, t_2)$ of degree $\leq j_1$ with respect to the first variable, and of degree $\leq j_2$ with respect to the second variable.

The proving method of (4.4) in [15] clearly indicates the straightforward way of its extension to the m-multiple case for any $m \geq 3$. Taking into account this observation, we conclude that if a function $f \in C(\mathbb{T}^m)$, then its Fourier series in (4.1) is summable H_q to $f(\boldsymbol{x})$ uniformly on $\mathbb{T}^m$, for any $0 < q < \infty$.

Now, applying the m-multiple version of Theorem 1.3 together with the extended version of Theorem 4.3 for functions $f \in C(\mathbb{T}^m)$, $m \in \mathbb{N}$, gives the following

Corollary 4.4. *If $f \in C(\mathbb{T}^m)$, then the m-multiple Fourier* *series of* f *in* (4.1) *statistically converges to* $f(\boldsymbol{x})$ *uniformly in* $\boldsymbol{x} \in \mathbb{T}^m$.

5. Tauberian theorems for statistical summability

We recall that a sequence (s_k) or a series with partial sums s_k, where the s_k are real or complex numbers, is said to be *summable* $(C, 1)$ to limit (or sum) s if

(5.1)

$$\lim_{n\to\infty} \sigma_n = s, \quad \text{where} \quad \sigma_n := (n+1)^{-1} \sum_{k=0}^{n} s_k, \ n = 0, 1, 2, \ldots .$$

is the *first arithmetic mean*, also called the Cesàro mean of first order. Clearly, the existence of the ordinary limit of a sequence (s_k) implies the existence of the ordinary limit of (σ_n), and the two limits coincide. The converse implication is not true in general. However, under certain supplementary condition(s), the implication

$$\lim_{n\to\infty} \sigma_n = s \quad \Rightarrow \quad \lim_{n\to\infty} s_k = s \tag{5.2}$$

does hold.

Such supplementary condition(s) under which the existence of ordinary limit follows from the existence of the limit by a summability method are called '*Tauberian*' one(s) with respect to the summability method in question. Likewise, such theorems containing condition(s) of this kind are also called '*Tauberian*' one(s), after *A. Tauber* (see, e.g., in [6, p. 149, lines 17-19 from above]), who first proved one of the simplest of them; namely, if the condition

$$\lim_{k\to\infty} ks_k = 0$$

is satisfied, then the implication (5.2) holds.

By Theorem 1.2, it is easy to see that if a sequence (s_k) is bounded, then the following implication holds:

$$\mathrm{st}-\lim_{k\to\infty} s_k = s \quad \Rightarrow \quad \mathrm{st}-\lim_{n\to\infty} \sigma_n = s.$$

Our goal in this Section is to present Tauberian conditions under which the converse implication holds true even without requiring the boundedness of the sequence (s_k):

$$\mathrm{st}-\lim_{n\to\infty} \sigma_n = s \quad \Rightarrow \quad \lim_{k\to\infty} s_k = s. \tag{5.3}$$

By Landau's definition (see in [9] and also in [6, pp. 124-126]), a sequence (s_k) of real numbers is said to be *slowly decreasing* (one may add to it: with respect to summability $(C,1)$) if

$$\liminf_{j\to\infty}(s_{k_j} - s_{n_j}) \geq 0 \tag{5.4}$$

whenever

$$n_j \to \infty \quad \text{and} \quad 1 \le k_j/n_j \to 1 \quad \text{as} \quad j \to \infty. \tag{5.5}$$

It is routine to check that (5.4)-(5.5) is equivalent to the following condition:

$$\lim_{\lambda \downarrow 1} \liminf_{n \to \infty} \min_{n < k \le \lambda n} (s_k - s_n) \ge 0;$$

and this latter one is satisfied if and only if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$ as close to 1 as we want, such that

$$s_k - s_n \ge -\varepsilon \quad \text{whenever} \quad n_0 \le n < k \le \lambda_0 n.$$

By Hardy's definition (see in [5] and also in [6, pp. 124-125], a sequence (s_k) of *complex numbers* is said to be *slowly oscillating* (one may add to it: with respect to summability $(C, 1)$) if

$$\lim_{j \to \infty} (s_{k_j} - s_{n_j}) = 0 \tag{5.6}$$

whenever the conditions in (5.5) are satisfied.

It is well known (see, e.g., in [6, p. 121]) that if a sequence (s_k) of real numbers satisfies Landau's *one-sided condition*:

(5.7)

$$k(s_k - s_{k-1}) \ge -H \quad \text{for some} \quad H > 0 \quad \text{and every} \quad k \ge 1,$$

then (s_k) is slowly decreasing. Likewise, if a sequence (s_k) of complex numbers satisfies Hardy's *two-sided Tauberian condition*:

$$(5.8) \qquad k|s_k - s_{k-1}| \le H \quad \text{for some} \quad H \quad \text{and every} \quad k,$$

then (s_k) is slowly oscillating.

The next two theorems proved in [12] by the present author give Tauberian conditions under which the implication (5.3) holds true.

Theorem 5.1. *If a sequence (s_k) of real numbers is statistically summable $(C, 1)$ to some s and it is slowly decreasing, then (s_k) converges to s.*

Theorem 5.2. *If a sequence (s_k) of complex numbers is statistically summable $(C, 1)$ to some s and it is slowly oscillating, then (s_k) converges to s.*

Next, we present analogous Tauberian theorems in the case of the so-called logarithmic summability, briefly: summability $(L, 1)$, of sequences (see, e.g., in [13], where the term 'harmonic summability' was used instead of 'logarithmic summability').

A sequence $(s_k) = (s_k : k = 1, 2, \ldots)$ of complex numbers or a series with partial sums s_k, is said to be *summable* $(L, 1)$

to s if

$$\lim_{n\to\infty} \tau_n = s,$$

where

$$\tau_n := \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{s_k}{k} \quad \text{and} \quad \ell_n := \sum_{k=1}^{n} \frac{1}{k} \sim \log n, \quad n = 1, 2, \ldots$$

(cf. (5.1)), and the logarithm is to the natural base e. It is easy to prove (see in [14]) that if a sequence (s_k) is summable $(C, 1)$ to some s, then it is summable $(L, 1)$ to the same s, but the converse implication is not true in general.

Next, we present Tauberian conditions under which the converse implication holds true even in the following more general form:

$$\text{st} - \lim_{n\to\infty} \tau_n = s \quad \Rightarrow \quad \lim_{k\to\infty} s_k = s. \tag{5.9}$$

To this effect, a sequence (s_k) of real numbers is said to be *slowly decreasing with respect to summability* $(L, 1)$ (see in [13]) if condition (5.4) is satisfied whenever

$$n_j \to \infty \quad \text{and} \quad 1 < (\log k_j)/(\log n_j) \to 1 \quad \text{as} \quad j \to \infty \tag{5.10}$$

(cf. (5.5)).

Furthermore, a sequence (s_k) of complex numbers is said to be *slowly oscillating with respect to summability* $(L, 1)$ (see in [8] and also in [13]) if condition (5.6) is satisfied whenever condition (5.10) is satisfied.

It is easy to check that if a sequence (s_k) of real numbers satisfies the following *one-sided Tauberian condition*:

$$k(\log k)(s_k - s_{k-1}) \geq -H \quad \text{for some} \quad H > 0 \quad \text{and every} \quad k \geq 2$$

(cf. (5.7)), then (s_k) is slowly decreasing with respect to summability $(L, 1)$.

Likewise, if a sequence (s_k) of complex numbers satisfies the following *two-sided Tauberian condition*:

$$k(\log k)|s_k - s_{k-1}| \leq H \quad \text{for some} \quad H \quad \text{and every} \quad k \geq 2$$

(cf. (5.8)), then (s_k) is slowly oscillating with respect to summability $(L, 1)$.

The next two theorems give Tauberian conditions under which the implication (5.9) holds true.

Theorem 5.3. (see in [13, Theorem 1]). *If a sequence* (s_k) *of real numbers is statistically summable* $(L, 1)$ *to some* s *and it is slowly decreasing with respect to summability* $(L, 1)$*, then* (s_k) *converges to* s.

Theorem 5.4. (see in [13, Theorem 2]). *If a sequence* (s_k) *of complex numbers is statistically summable* $(L, 1)$ *to some* s *and it is slowly oscillating with respect to summability* $(L, 1)$*, then* (s_k) *converges to* s.

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