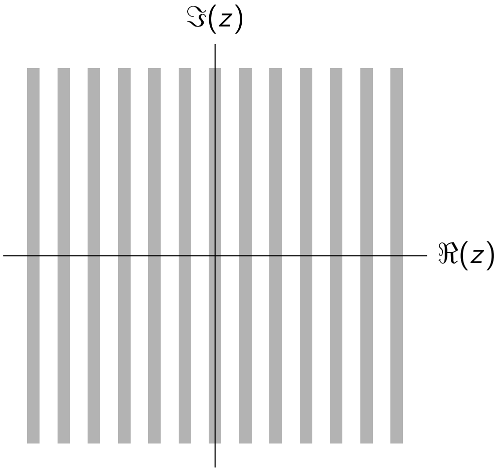
The Pyjama Problem

Freddie Manners

University of Oxford

August 30, 2013

The pyjama stripe



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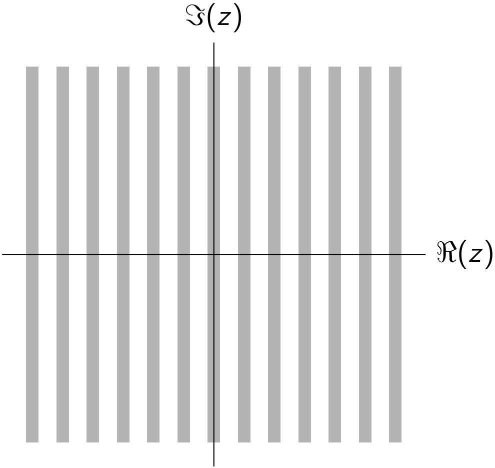

For $\varepsilon > 0$, the "pyjama stripe" is:

$$E = E(\varepsilon) := \{z \in \mathbb{C} : \Re(z) \pmod 1 \in (-\varepsilon, \varepsilon)\}$$

The pyjama problem

Question (Iosevich, Kolountzakis, Matolcsi 2006)

Can finitely many rotations of $E(\varepsilon)$ about the origin cover the plane?

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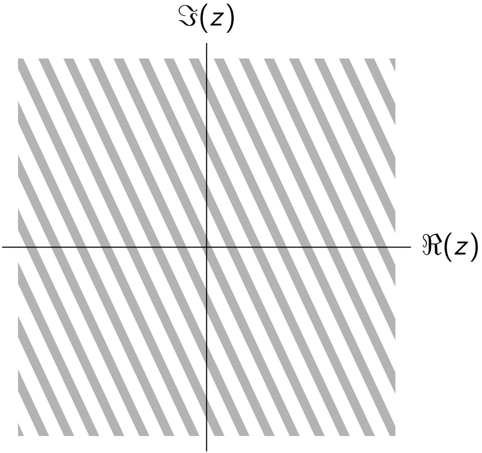

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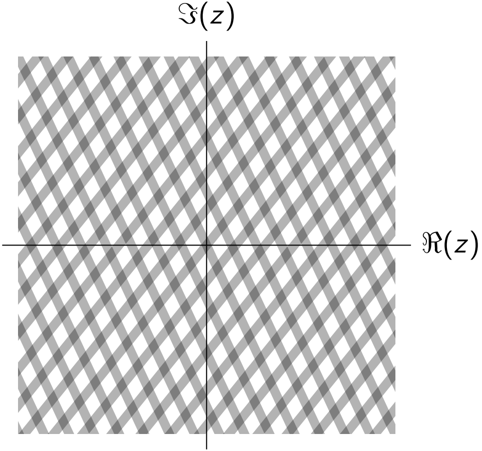

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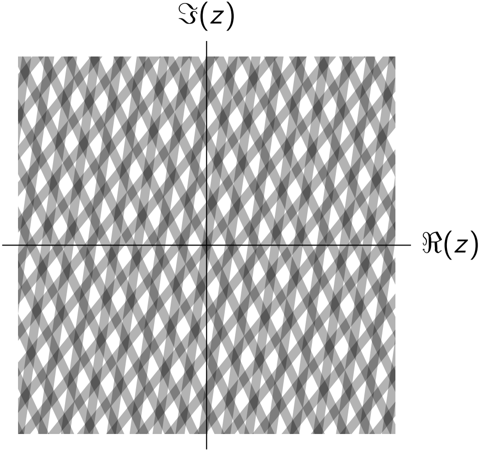

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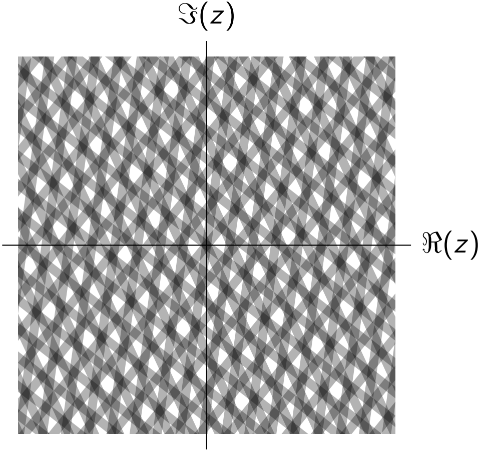

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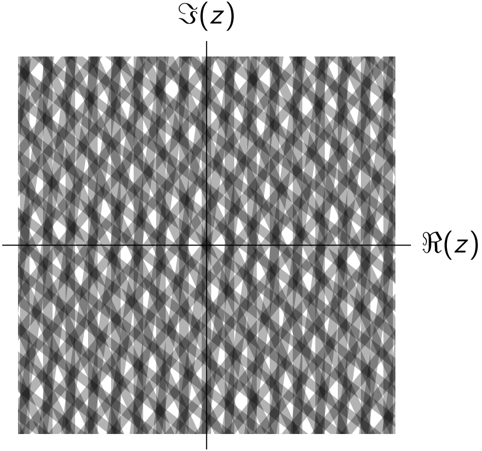

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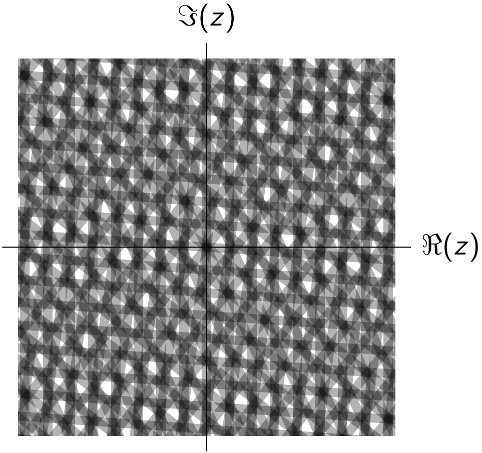

Easy results

Write $\varepsilon_{\mathrm{min}}$ for the infimum of ε for which the answer is yes.

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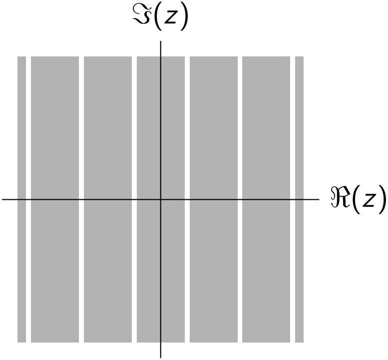

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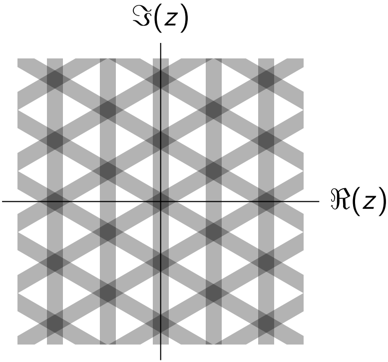

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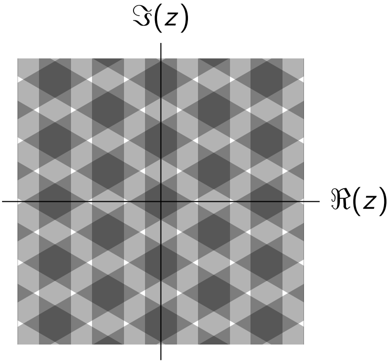

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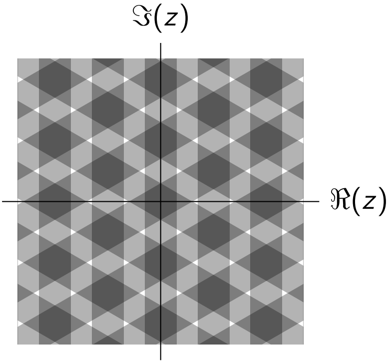

Improving on $1/3$ is hard!

Results

Theorem (Malikiosis, Matolcsi, Ruzsa 2012)

$\varepsilon_{\mathrm{min}} \leq 1/3 - 1/48.$

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Theorem (M, 2013)

$\varepsilon_{\mathrm{min}} = 0$.

Random constructions don't work

For $\theta \in \mathbb{C}$, $|\theta| = 1$ write

$$E_\theta = \theta^{-1} E = \{ z \in \mathbb{C} : \Re(\theta z) \mod 1 \in (-\varepsilon, \varepsilon) \} .$$

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A point $x \in \mathbb{R}$ is uncovered if and only if

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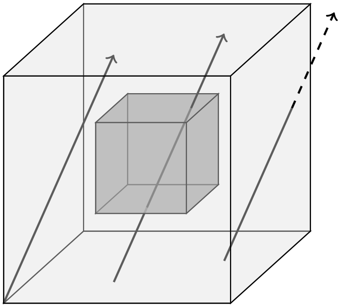

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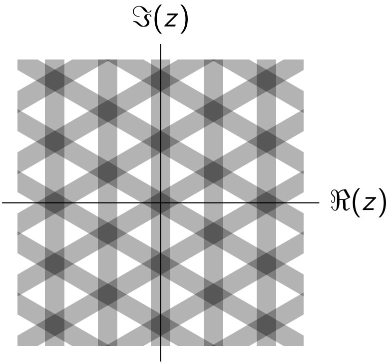

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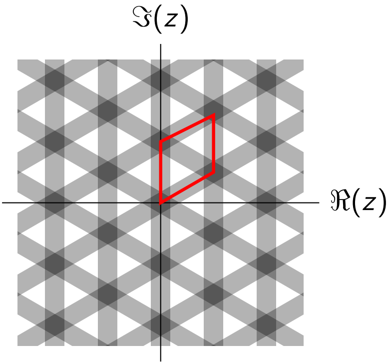

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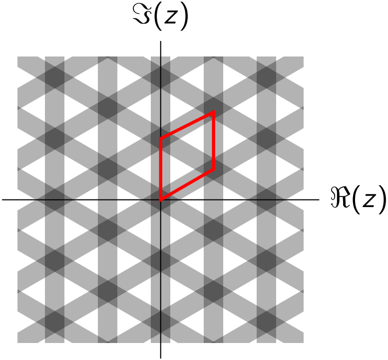

Lemma (Malikiosis, Matolcsi, Ruzsa 2012)

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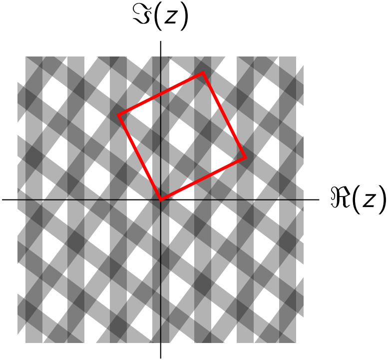

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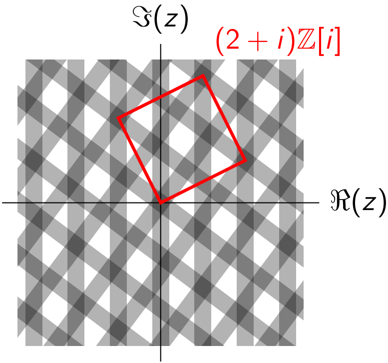

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$D\mathbb{Z}[i]$-periodic $\iff$ $\Theta \subseteq \frac{1}{D}\mathbb{Z}[i]$.

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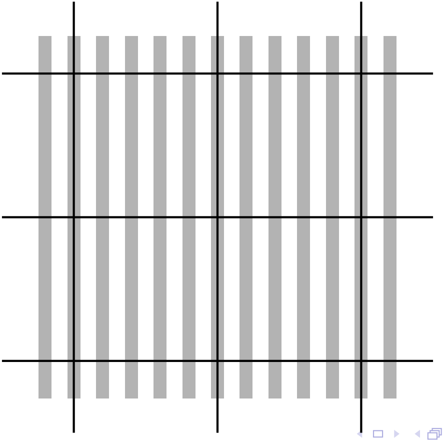

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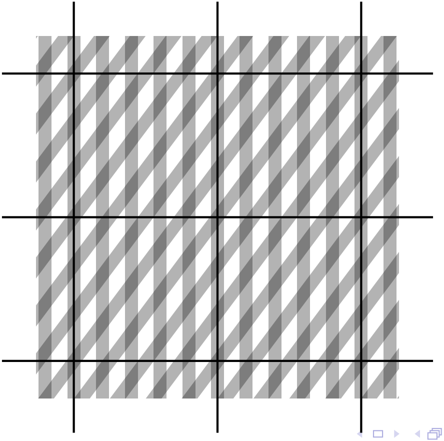

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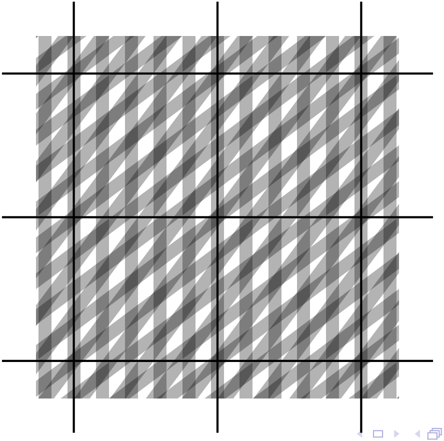

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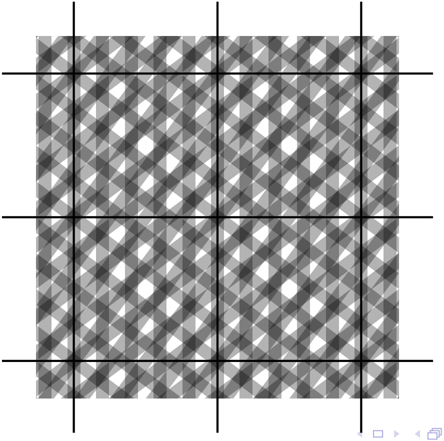

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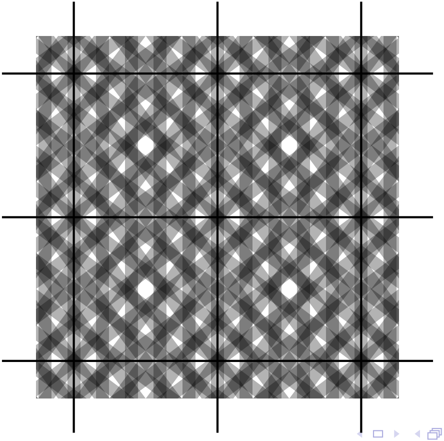

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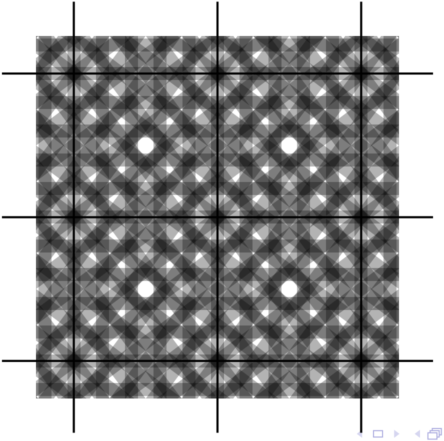

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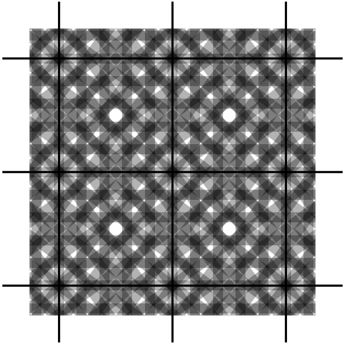

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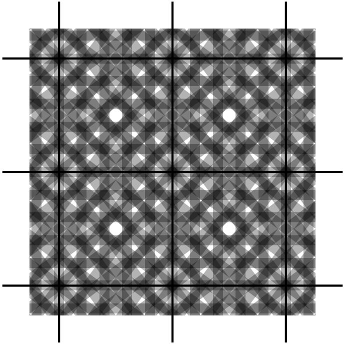

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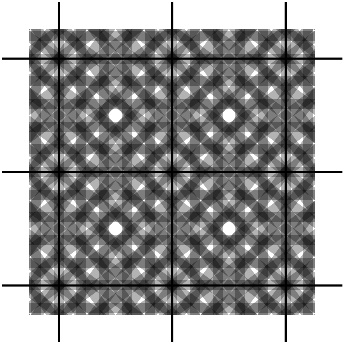

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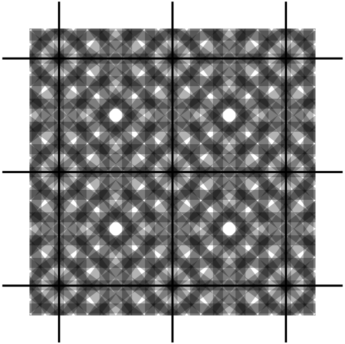

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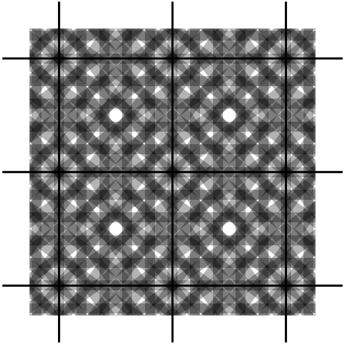

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First in an infinite family of *rational obstructions*.

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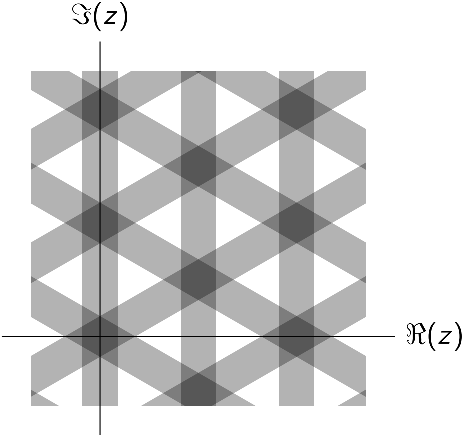

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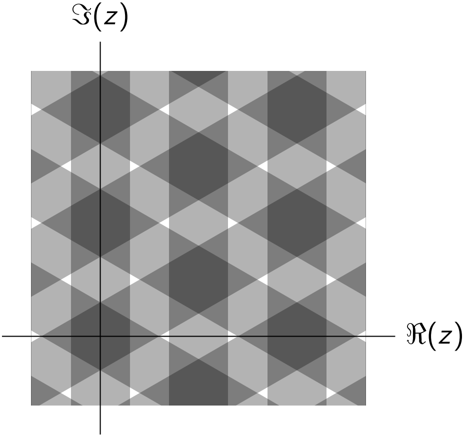

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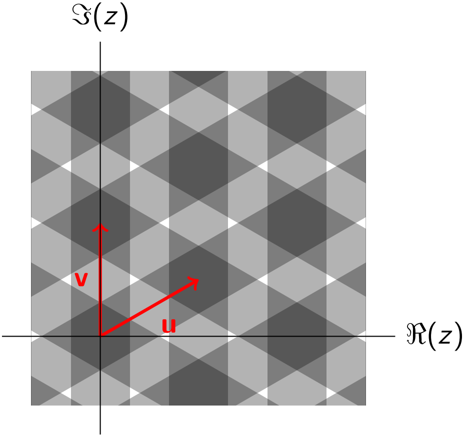

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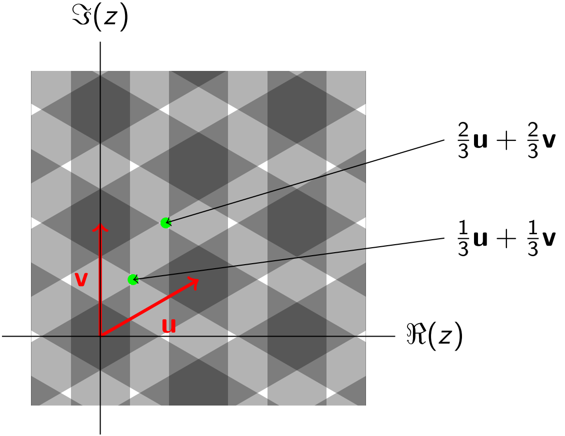

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Bad points are all near $\left(\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}\right)$ or $\left(\frac{2}{3}\mathbf{u} + \frac{2}{3}\mathbf{v}\right) \pmod{\Lambda}$.

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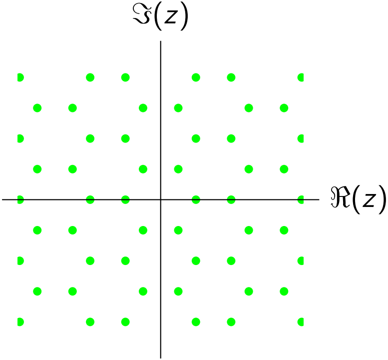

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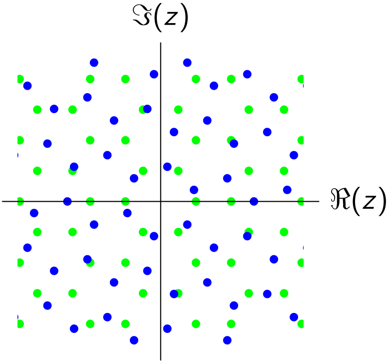

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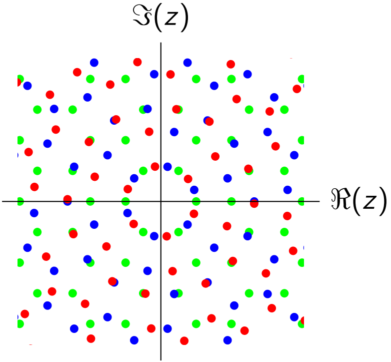

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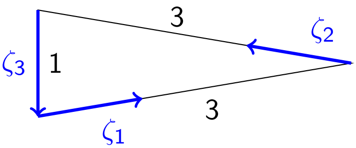
Choose ζ_1, ζ_2, ζ_3 in the unit circle such that $3\zeta_1 + 3\zeta_2 + \zeta_3 = 0$.



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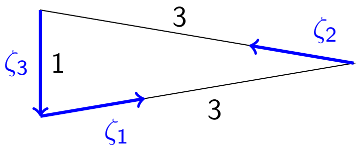

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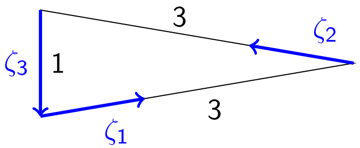

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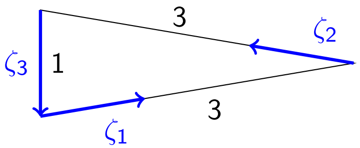
$z \in \mathbb{C}$ is uncovered overall $\iff$ $\zeta_1 z, \zeta_2 z, \zeta_3 z$ are all bad points.

Beating $\varepsilon = 1/3$

Bad points are all near $\left(\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}\right)$ or $\left(\frac{2}{3}\mathbf{u} + \frac{2}{3}\mathbf{v}\right) \pmod{\Lambda}$.

If $x, y \in \mathbb{C}$ are bad, then (for example) $-(3x + 3y)$ is not.

Choose ζ_1, ζ_2, ζ_3 in the unit circle such that $3\zeta_1 + 3\zeta_2 + \zeta_3 = 0$.



Rotate the triangle configuration by each of $\zeta_1^{-1}, \zeta_2^{-1}, \zeta_3^{-1}$.

$z \in \mathbb{C}$ is uncovered overall $\iff$ $\zeta_1 z, \zeta_2 z, \zeta_3 z$ are all bad points.

But, $\zeta_3 z = -(3\zeta_1 z + 3\zeta_2 z)$; $\Rightarrow\Leftarrow$.

The strategy

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– Hope that "rational obstructions" are all that goes wrong with periodic constructions.

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- Hope that "rational obstructions" are all that goes wrong with periodic constructions.
- Deal with the remaining bad points using a variant of the $3\zeta_1 + 3\zeta_2 + \zeta_3 = 0$ trick.